Canonical Quantization of the Boundary Wess-Zumino-Witten Model

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Abstract

We present an analysis of the canonical structure of the Wess-Zumino-Witten theory with untwisted conformal boundary conditions. The phase space of the boundary theory on a strip is shown to coincide with the phase space of the Chern-Simons theory on a solid cylinder (a disc times a line) with two Wilson lines. This reveals a new aspect of the relation between two-dimensional boundary conformal field theories and three-dimensional topological theories. A decomposition of the Chern-Simons phase space on a punctured disc in terms of the one on a punctured sphere and of coadjoint orbits of the loop group easily lends itself to quantization. It results in a description of the quantum boundary degrees of freedom in the WZW model by invariant tensors in a triple product of quantum group representations. The bulk primary fields of the WZW model are shown to combine, in the action on the space of states of the boundary theory, the usual vertex operators of the current algebra with monodromy acting on the quantum group invariant tensors. We present the details of this construction for the spin 1/2 fields in the $SU(2)$ WZW theory, establishing their locality and computing their 1-point functions.

1 Introduction

Two-dimensional boundary conformal field theory is a subject under intense study. Models of the theory find multiple applications in the analysis of two- or 1+1-dimensional critical phenomena in the presence of physical boundaries [9], localized impurities in a metal [1], or point contacts in quantum Hall devices or quantum wires [19]. In string theory they describe branes on which open strings end [32, 33]. A boundary conformal field theory model is a quantum field theory in a half space that exhibits invariance under the conformal transformations preserving the boundary. In two dimensions such transformations form an infinite dimensional group of reparametrizations of a line. This rich symmetry (or its generalizations) are powerful enough to allow in many cases a classification of possible solutions, similarly as in the simpler case without boundary [6]. Although much progress has been achieved in understanding boundary CFT's since the seminal paper of Cardy [9], much more remains to be done. The structure involved in the boundary CFT's is richer than in the bulk theory and classification program involves new notions [31, 39]. One approach that offered a conceptual insight into the properties of correlation functions of boundary conformal models consisted of relating them to boundary states in three-dimensional topological field theories [16, 17]. In the simplest case of the boundary Wess-Zumino-Witten (WZW) models (conformal sigma models with a group G as a target [37]), the topological three-dimensional model appears to be the group G Chern-Simons (CS) gauge theory [38].

The purpose of the present paper is to demonstrate another facet of the relationship between the boundary WZW models and the CS theory, already present at the classical level. We shall discover it by analyzing the structure of the phase space of the WZW model with the most symmetric boundary conditions. These, so called "untwisted", boundary conditions restrict the boundary values of the classical fields of the model to fixed conjugacy classes in G which are labeled by weights of the Lie algebra $\mathbf{g}$ of G. Such boundary conditions reduce to the Dirichlet conditions for toroidal targets. We shall show that the phase space of the WZW model on a strip with the untwisted boundary conditions is isomorphic to the phase space of the CS theory on a disc D times the time line $\mathbf{R}$, with two timelike Wilson lines corresponding to the weights labeling the boundary conditions. This generalizes the case with one Wilson line which is well known to reproduce the coadjoint orbits of the (central extension) of the loop group LG [13]. The isomorphism to the CS theory on $D \times \mathbf{R}$ is another manifestation of the chiral character of the boundary CFT which has half of the bulk symmetries and correlation functions given by special chiral conformal blocks on a double surface. The CS theory (certainely abelian but possibly nonabelian) describes the long range degrees of freedom in the physics of Quantum Hall Effect [20, 21, 8], with Wilson lines representing excited Laughlin vortices. Since the disc geometry appears naturally in material samples, our identification raises a possibility of new applications of boundary CFT to condensed matter physics.

The phase space of the CS theory on $D \times \mathbf{R}$ with two Wilson lines may be decomposed in terms of the phase space of the CS theory on $S^2 \times \mathbf{R}$ with three Wilson lines and the coadjoint orbits of the loop group, with one Wilson line indexed by the same weight as the loop group orbit. This is the realization of the phase space of the boundary WZW model that we analyze in detail[1]. The symplectic structure of the CS theory on $\Sigma \times \mathbf{R}$, where Σ is a compact surface without boundary, with timelike Wilson lines, has been studied in a number of mathematical papers, see e.g. [26, 27, 28]. The phase space of the theory is composed of flat connections on punctured Σ, modulo gauge transformations, with prescribed conjugacy classes of the holonomy around the punctures. We shall make use of the paper [2] that contains the calculation of the symplectic structure of the phase space in terms of the holonomy of the flat connection. This presentation of the CS phase spaces allows us to identify a factor in the phase space of the boundary WZW model as the CS phase space for the $S^2 \times \mathbf{R}$ geometry with three timelike Wilson lines. The latter space, as was realized in [2], may be also described in terms of the Poisson-Lie geometry. It is isomorphic to a reduction of a product of symplectic leaves of the Poisson-Lie group G^* dual to G equipped with the r-matrix Poisson-Lie group structure [34]. One reduces the product of the leaves with respect to the diagonal Poisson-Lie "dressing" action of G.

The above identifications permit a decomposition of the phase space of the boundary WZW model in terms of the coadjoint orbits of LG and of the reduced products of symplectic leaves of G^*. The main point of the above analysis is that in such a presentation the WZW phase space may be easily quantized. The coadjoint orbits of the loop group give rise upon quantization to the unitary projective representations of LG (or of the corresponding affine current algebra $\widehat{\mathbf{g}}$). Geometric quantization of the phase-space of the CS theory on $S^2 \times \mathbf{R}$ with Wilson lines produces the space of conformal blocks of the WZW theory on punctured S^2. As for the symplectic leaves of G^*, they may be quantized to irreducible representations of the quantum deformation $\mathcal{U}_q(\mathbf{g})$ of the enveloping algebra of $\mathbf{g}$. The diagonal reduction of the product of symplectic leaves imposes on the quantum level a restriction to invariant tensors of the product of quantum group representations. It is indeed well known that the conformal blocks of the WZW model on a punctured sphere may be identified with ("good") invariant tensors of the quantum group [18]. This is, in fact, the way by which the quantum group tensors entered the analysis of bulk CFT's. Their appearance in the boundary theory is even more natural as in the latter they describe directly a part of the physical degrees of freedom.

A concrete realization of the space of quantum states of the boundary WZW model in geometric terms would not be very useful if it did not lead to a natural description of the rest of the quantum field theory structure. We then show how to use our geometric approach to construct the action of the bulk primary fields in the Hilbert space of the boundary model. The bulk operators are built by combining the vertex operators acting between the unitary representations of the current algebra $\widehat{\mathbf{g}}$ with "monodromy" expressed as a combination of quantum group generators and intertwiners that acts in the spaces of invariant quantum group tensors. We make this construction explicit for the case of the $SU(2)$ group and spin $1/2$ bulk fields using free field realizations of the current algebra and of the quantum group representations. The main result here is the proof of locality of the constructed fields. Our analysis does not exhaust the algebraic content of the boundary WZW $SU(2)$ model. We do not discuss the higher spin bulk operators (they could be constructed along similar lines as for the spin $1/2$ fields or by fusing the latter). Neither do we discuss the boundary operators, although the ones which do not change boundary

[1] A direct discussion of the CS theory on $D \times \mathbf{R}$ with two Wilson lines will be presented elsewhere.

conditions may be easily obtained from the bulk operators by sending the insertion point to the boundary. An extension of the present approach to boundary changing operators would require going beyond the strip geometry of the world-sheet analyzed here. Other obvious open problems are an extension of the analysis to twisted boundary conditions [33] and to other groups.

The paper is organized as follows. In Sect. 2, we describe the canonical structure of the bulk WZW theory studied in numerous publications, see [14, 4, 12, 11, 22, 5]. Our exposition follows closely that of [23, 15]. In particular, we analyze the decomposition of the bulk phase space into chiral components. In Sect. 3, we describe the phase space of the boundary WZW model stressing the similarities and differences with the chiral sector of the bulk theory. In Sect. 4.1, we recall the results of [2] about the phase space of the CS theory on $S^2 \times \mathbf{R}$ with three Wilson lines and identify the latter space with the phase space of the boundary degrees of freedom in the boundary WZW model. In Sect. 4.2, we show how to identify the complete phase space of the boundary theory with the phase space of the CS theory on $D \times \mathbf{R}$ with two Wilson lines. Sect. 5 recalls the relations between the CS phase space and Poisson-Lie symplectic leaves following again the results of [2]. Sect. 6 discusses quantization of the building blocks of the boundary theory. In Sect. 6.1, we describe the Hilbert space of states in the boundary theory that factors into the unitary representations of the current algebra and the finite-dimensional spaces of 3-point conformal blocks. In Sect. 6.2, we recall the free field realizations of the unitary representations of the current algebra $\widehat{su}(2)$ [36] and of the spin 1/2 vertex operators [7]. Sect. 6.3 is devoted to similar constructions for the $\mathcal{U}_q(su(2))$ quantum group [15]. We obtain a "free field" realization of the spaces of invariant quantum group tensors and of the action on it of the monodromy operators. In Sect. 7, we make use of the preceding constructions to describe the action of the bulk spin 1/2 fields in the Hilbert space of the boundary theory. We define the quantum bulk fields in Sect. 7.1 and check their locality in Sects. 7.2 and 7.3. Finally, in Sect. 7.4, we compute some simple matrix elements of these operators. Appendices establish two algebraic identities used in the text.

2 Canonical quantization of the bulk WZW model

On the classical level, the WZW model is specified by the action functional of classical fields. Its symmetry structure is identified by examining field transformations mapping classical solutions (i.e. extremal points of the action) to classical solutions. Quantization of the model is performed in the way that preserves the classical symmetries.

Let us start by reminding how this is done for the WZW model in the bulk, see [23, 15]. As the two-dimensional (Minkowski) space-time M we shall take the cylinder $\mathbf{R} \times S^1$ with the coordinates $(t, x \bmod 2\pi)$. We shall also use the light-cone coordinates $x^\pm = x \pm t$ on M in which the metric takes the form $ds^2 = dx^+ dx^-$. The fields of the WZW model on M are the maps $g : M \to G$, see Fig. 1,

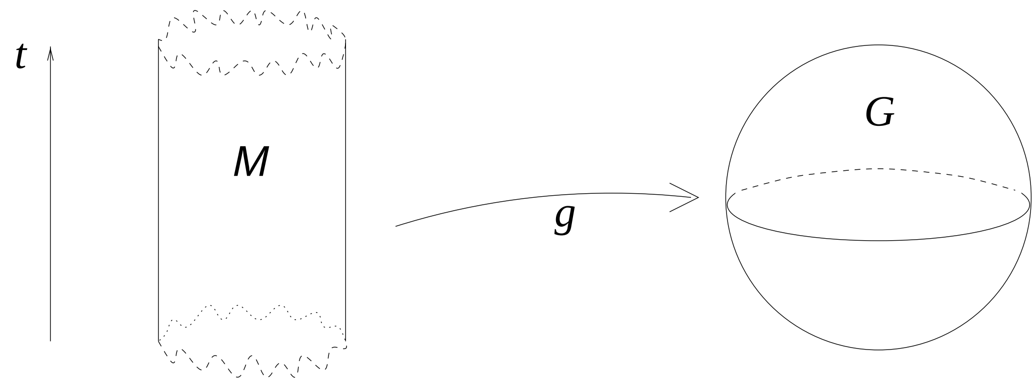

Fig. 1

where G is a compact group that we shall take simple, connected and simply connected. The action of the model is given by the expression

$$S(g) \;=\; \frac{k}{4\pi}\int\limits_M [\,\mathrm{tr}\,(g^{-1}\partial_+ g)(g^{-1}\partial_- g)\; dx^+ dx^- \;+\; g^*\omega\,]\,, \tag{1}$$

where k is the coupling constant (the "level" of the model), tr stands for the Killing form on the (complexification of the) Lie algebra $\mathbf{g}$ of G (normalized to give length square 2 of the long roots), ω is a 2-form on G satisfying

$$d\omega(g) \;=\; \tfrac{1}{3}\;\mathrm{tr}\,(g^{-1}dg)^3 \;\equiv\; \theta(g) \tag{2}$$

and $g^*\omega$ denotes the pull-back of ω (we use the symbol g to denote both the field mapping M to G and an element of the group G). The last equality requires a comment, since the 3-form θ on G is closed but not exact so that no 2-form ω satisfying Eq. (2) exists globally on G. To simplify the more complex story (see e.g. Sect. 7 of [24]), we shall assume that the values of the field g belong to an open subset of G on which one can define such a 2-form. On the space-time without boundary, local variations of the action and, consequently, also the classical equations, will be independent of the choice of ω.

It will be convenient to rewrite the action (1) in the first order formalism, see [23]. To this end, we introduce additional Lie-algebra valued coordinates $\xi_\pm$ (which will represent the values of field derivatives) and we define a 2-form α on the extended space $P \equiv M \times G \times \mathbf{g}^2$,

$$\alpha \;=\; \frac{k}{4\pi}\,[\,\mathrm{tr}\,\xi_+\xi_-\; dx^+dx^- \;-\; i\,\mathrm{tr}\,\xi_+(g^{-1}dg)\,dx^+ \;+\; i\,\mathrm{tr}\,\xi_-(g^{-1}dg)\,dx^- \;+\; \omega(g)\,]\,. \tag{3}$$

The first order action takes the simple form of the space-time integral of a pull-back of the form α:

$$S(\Phi) \;=\; \int\limits_M \Phi^*\alpha\,, \tag{4}$$

where $\Phi = (I, g, \xi_+, \xi_-)$ maps the space-time to P (I stands here for the identity map of M). If the new fields $\xi_\pm$ are given by the light-cone derivatives of g,

$$\xi_\pm \;=\; \tfrac{1}{i}\, g^{-1}\partial_\pm g\,, \tag{5}$$

then the first order action (4) reduces to the original expression (1). The first order formalism is, however, more geometric. For example, the variation of the action (4) takes the form

$$\delta S(\Phi) \;=\; \int\limits_M \Phi^*(\iota_{\delta\Phi}\, d\alpha)\,, \tag{6}$$

where $\iota_{\delta\Phi}$ denotes the interior product (contraction) with the vector field $\delta\Phi$ giving the infinitesimal variation of Φ ($\delta\Phi$ is defined on the range of Φ and is tangent to P). Consequently, the classical equations in the first order formalism take the form

$$\Phi^*(\iota_X\, d\alpha) \;=\; 0 \qquad \text{for every vector field } X \text{ on } P. \tag{7}$$

These equations are equivalent to the relations (5) supplemented with the variational equation $\delta S(g) = 0$ for the second-order action. The latter requires that

$$\partial_-(g\,\partial_+ g^{-1}) \;=\; 0\,, \tag{8}$$

or, equivalently, that $\partial_+(g^{-1}\partial_- g) = 0$.

Eq. (8) is easy to solve. The solutions on the cylinder M decompose into the left- and right-moving components (generalizing the decomposition of the solutions of the linear wave equation):

$$g(t,x) \;=\; g_L(x^+)\; g_R(x^-)^{-1}\,, \tag{9}$$

where the chiral fields $g_{L,R}$ are arbitrary G-valued maps on the real line satisfying

$$g_{L,R}(y+2\pi) \;=\; g_{L,R}(y)\,\gamma \tag{10}$$

with the same monodromy $\gamma \in G$. By the right multiplication of g_L and g_R by the same element of G, a change that does not effect the solution, one may reduce the monodromy γ to the Cartan subgroup $T \subset G$ or, even more, to the form $\gamma = \mathrm{e}^{2\pi i\tau}$, where τ belongs to the positive Weyl alcove $\mathcal{A}_W$ in the Cartan algebra $\mathbf{t}$. Nevertheless, it will be sometimes convenient to work with general γ.

The space $\mathcal{P}$ of the classical solutions given explicitly by Eq. (9) forms the phase space of the WZW model on the cylinder. As usual, the phase space comes with the canonical symplectic structure. The symplectic form Ω on $\mathcal{P}$ may be conveniently expressed in the first order formalism, see e.g. [23]. Namely,

$$\Omega(\delta_1\Phi,\, \delta_2\Phi) \;=\; \int\limits_{M_t} \Phi^*(\iota_{\delta_2\Phi}\, \iota_{\delta_1\Phi}\, d\alpha)\,, \tag{11}$$

where M_t denotes the constant time section of M. The integral on the right hand side is t-independent since the integrated form is closed. Explicitly [23]:

$$\Omega \;=\; \frac{k}{4\pi}\int_0^{2\pi}\mathrm{tr}\,\left[-\delta(g^{-1}\partial_t g)\,g^{-1}\delta g\;+\;2\,(g^{-1}\partial_+ g)\,(g^{-1}\delta g)^2\right]\,dx\,, \tag{12}$$

where δ denotes here the exterior derivative on $\mathcal{P}$ and the x-integral is performed with fixed t.

Although we have originally assumed that the group G was compact, in all the formulae above, we could replace G by its complexification. The phase space $\mathcal{P}$ would then become a complex symplectic manifold. Below, we shall work in the complex context whenever more convenient.

The symplectic structure of $\mathcal{P}$ allows to assign to functions $\mathcal{F}$ on $\mathcal{P}$ the Hamiltonian vector fields $\mathcal{X}_{\mathcal{F}}$ such that $d\mathcal{F}=\iota_{\mathcal{X}_{\mathcal{F}}}\Omega$ and to define the Poisson bracket $\{\mathcal{F},\mathcal{F}'\}=\mathcal{X}_{\mathcal{F}}(\mathcal{F}')$ of functions on the phase space. Some equal-time Poisson brackets are easy to compute. For example, if $g(t,x)_1$ and $g(t,x)_2$ denote the matrices $g(t,x)\otimes I$ and $I\otimes g(t,x)$ in a fixed representation of G, with a similar notation for the Lie-algebra valued fields, then

$$\begin{aligned}
\Big\{\,g(t,x)_1\,,\;g(t,x')_2\Big\} \quad&=\;0\,,\\
\Big\{\,g(t,x)_1\,,\;(g^{-1}\partial_t g)(t,x')_2\Big\} \quad&=\;-\tfrac{4\pi}{k}\,\delta(x-x')\;g(t,x)_1\;C_{12}\,,\\
\Big\{\,(g^{-1}\partial_t g)(t,x)_1\,,\;(g^{-1}\partial_t g)(t,x')_2\Big\} \;&=\;\tfrac{8\pi}{k}\,\delta(x-x')\,\left[\,C_{12}\,,\;(g^{-1}\partial_+ g)(t,x)_1\,\right],
\end{aligned} \tag{13}$$

where the matrix product is implied on the left hand side and C_{12} stands for the matrix representing the Casimir element $\sum t^a\otimes t^a\;\in\;\mathbf{g}\otimes\mathbf{g}$, with the generators t^a of the Lie algebra $\mathbf{g}$ such that $\mathrm{tr}\,t^at^b=\delta^{ab}$.

It is easy to identify the symmetry structure of the WZW theory on the cylinder. First, the loop group LG composed of the periodic maps h from the line to G with period 2π act on the phase space $\mathcal{P}$ in two ways by

$$g(x^+,x^-)\;\longmapsto\;h(x^+)\,g(x^+,x^-)\,,\qquad\qquad g(x^+,x^-)\;\longmapsto\;g(x^+,x^-)\,h(x^-)^{-1} \tag{14}$$

preserving the symplectic structure. On the infinitesimal level, these actions are generated by the currents[2]

$$J_{_L}\;=\;ik\,g\partial_+g^{-1}\;=\;ik\,g_{_L}\partial_+g_{_L}^{-1}\,,\qquad -J_{_R}\;=\;-ik\,g^{-1}\partial_-g\;=\;-ik\,g_{_R}\partial_-g_{_R}^{-1} \tag{15}$$

which are periodic functions with period 2π of x^+ and x^-, respectively. Second, there are two commuting actions on $\mathcal{P}$ of the group $Diff_+(S^1)$ of the orientation-preserving diffeomorphisms D of the circle $S^1\cong\mathbf{R}/2\pi\mathbf{Z}$ given by:

$$g(x^+,x^-)\;\longmapsto\;g(D^{-1}(x^+),\,x^-)\,,\qquad\qquad g(x^+,x^-)\;\longmapsto\;g(x^+,\,D^{-1}(x^-))\,. \tag{16}$$

[2] More precisely, the functions $\mathcal{F}\;=\;\pm\frac{1}{2\pi}\int_0^{2\pi}\mathrm{tr}\;\delta\Lambda(x^\pm)\,J_{_{L,R}}(t,x)\,dx$ generate the Hamiltonian vector fields corresponding to the action of the loop group elements $h(x^\pm)=\mathrm{e}^{-i\,\delta\Lambda(x^\pm)}$.

They also preserve the symplectic structure. Their infinitesimal versions are generated by the non-vanishing components of the energy-momentum tensor[3]

$$T_L \;=\; -\frac{k}{2}\,\mathrm{tr}\,(g\,\partial_+ g^{-1})^2 \;=\; \frac{1}{2k}\,\mathrm{tr}\;J_L^2\,, \qquad -T_R \;=\; \frac{k}{2}\,\mathrm{tr}\,(g^{-1}\partial_- g)^2 \;=\; -\frac{1}{2k}\,\mathrm{tr}\;J_R^2\,. \tag{17}$$

These are the infinite-dimensional symmetries of the theory.

In order to achieve a formulation of the classical WZW model that lends itself more easily to quantization, it is convenient to express the symplectic structure of the phase space $\mathcal{P}$ in terms of the chiral components $g_{L,R}$ of the classical solutions. One obtains:

$$\Omega \;=\; \Omega_L \;-\; \Omega_R \tag{18}$$

where

$$\Omega_L \;=\; \frac{k}{4\pi}\Big[\int_0^{2\pi}\mathrm{tr}\;(g_L^{-1}\delta g_L)\,\partial_x(g_L^{-1}\delta g_L)\,dx \;\;+\;\; \mathrm{tr}\;(g_L^{-1}\delta g_L)(0)\,(\delta\gamma)\,\gamma^{-1}\Big] \tag{19}$$

and Ω_R is given by the same formula with g_R replacing g_L. The reversed sign in front of Ω_R is the source of the negative signs in front of J_R and of T_R above. The chiral 2-forms $\Omega_{L,R}$ on $\mathcal{P}$ are not closed. An easy computation gives:

$$\delta\Omega_L \;=\; \frac{k}{4\pi}\,\theta(\gamma)\,, \tag{20}$$

where, as before, $\theta(\gamma) \,=\, \frac{1}{3}\,\mathrm{tr}\,(\gamma^{-1}d\gamma)^3$. If we restrict, however, the monodromy of the twisted-periodic fields g_L to be of the form $\gamma = \mathrm{e}^{2\pi i\tau}$ with $\tau \in \mathcal{A}_W$ then the forms $\Omega_{L,R}$ become closed and define the symplectic structure on the chiral components $\mathcal{P}_{L,R}$ of the phase space composed of fields g_L and g_R with the restricted monodromy.

One may also proceed differently [23] keeping the monodromies general and introducing modified forms

$$\widetilde{\Omega}_L \;=\; \Omega_L \;-\; \rho(\gamma)\,, \qquad \widetilde{\Omega}_R \;=\; \Omega_R \;-\; \rho(\gamma)\,, \tag{21}$$

where ρ is a 2-form on G. The decomposition $\Omega = \widetilde{\Omega}_L - \widetilde{\Omega}_R$ still holds since the ρ-terms cancel. If the form ρ were such that $d\rho = \theta$ then the modified 2-forms $\widetilde{\Omega}_{L,R}$ would be closed. Note that we recover for ρ the same condition as for the 2-form ω entering the action of the model, see Eq. (2). Of course, as before, that condition cannot be satisfied globally. In the complex setup, a convenient solution is to consider only generic monodromies that may be parametrized by the Gauss decomposition $\gamma = \gamma_-\gamma_+^{-1}$ with $\gamma_\pm$ in the Borel subgroups $B_\pm = N_\pm T \subset G$. $N_\pm$ denote the nilpotent subgroups of $B_\pm$ and γ_+ and γ_-^{-1} are taken with coinciding components in the Cartan subgroup T. The choice

$$\rho(\gamma) \;=\; \mathrm{tr}\,(\gamma_-^{-1}d\gamma_-)(\gamma_+^{-1}d\gamma_+)\,, \tag{22}$$

fulfills the condition $d\rho \,=\, \theta$ rendering the forms $\widetilde{\Omega}_{L,R}$ closed and providing symplectic structures on the spaces $\widetilde{\mathcal{P}}_{L,R}$ of chiral fields (with the monodromies parametrized by the Gauss decomposition).

[3] More precisely, $\mathcal{F} = \pm\frac{1}{2\pi}\int_0^{2\pi}\delta\xi(x^\pm)\,T_{L,R}(t,x)\,dx$ generate the Hamiltonian vector fields corresponding to the action of the diffeomorphisms $D = \mathrm{e}^{\delta\xi(x^\pm)\,\partial_\pm}$.

In [15] a further change of variables, a classical version of the so called vertex-IRF (interaction round the face) transformation, was described. It decomposed a chiral field into the product of a closed loop in G, a multi-valued field in the Cartan subgroup and a constant element in G:

$$g_{_L}(x) \;=\; h(x)\,\mathrm{e}^{i\tau x}\,g_{_0}^{-1} \;\equiv\; h_{_L}(x)\,g_{_0}^{-1}\,, \tag{23}$$

where $h \in LG$, τ belongs to the positive Weyl alcove $\mathcal{A}_W \subset \mathbf{t}$ (in the complex setup, $\mathcal{A}_W$ should admit arbitrary imaginary parts of τ) and $g_{_0} \in G$. For the monodromy of $g_{_L}$, we obtain

$$\gamma \;=\; g_{_0}\,\mathrm{e}^{2\pi i\tau}\,g_{_0}^{-1}\,. \tag{24}$$

The parametrization (23) induces the following decomposition of the form $\widetilde{\Omega}_{_L}$:

$$\begin{aligned}\widetilde{\Omega}_{_L} \;&=\; \tfrac{k}{4\pi}\int_0^{2\pi}\mathrm{tr}\;[(h^{-1}\delta h)\,\partial_x(h^{-1}\delta h) \,+\, 2i\tau\,(h^{-1}\delta h)^2 \,-\, 2i\,(\delta\tau)(h^{-1}\delta h)]\,dx\\
&\quad +\tfrac{k}{4\pi}\,\mathrm{tr}\,(g_{_0}^{-1}\delta g_{_0})\,\mathrm{e}^{2\pi i\tau}\,(g_{_0}^{-1}\delta g_{_0})\,\mathrm{e}^{-2\pi i\tau} \,+\, k\,i\,\mathrm{tr}\,(\delta\tau)(g_{_0}^{-1}\delta g_{_0}) \,-\, \tfrac{k}{4\pi}\,\rho(g_{_0}\,\mathrm{e}^{2\pi i\tau}\,g_{_0}^{-1})\\
&\equiv\; \Omega^{LG} \,+\, \Omega^{PL}\,.\end{aligned} \tag{25}$$

The 2-form Ω^{LG} coincides with the restriction of the chiral 2-form $\Omega_{_L}$ to the subspace $\mathcal{P}_{_L}$ of fields $h_{_L}$ with monodromy $\gamma = \mathrm{e}^{2\pi i\tau}$. The symplectic space $\mathcal{P}_{_L}$ may be identified with the so called "model space" $\mathcal{M}_{_{LG}} = LG \times \mathcal{A}_W$ of the loop group, a symplectic space roughly speaking containing once each coadjoint orbit $\mathcal{O}_{_{LG}}(\tau)$ passing through τ of the (central extension $\widehat{LG}$ of the) loop group. More precisely, for fixed τ, Ω^{LG} gives the pull-back to LG of the Kirillov symplectic form Ω^{LG}_τ on $\mathcal{O}_{_{LG}}(\tau)$. We infer that

$$\mathcal{P}_{_L} \;\cong\; \mathcal{M}_{_{LG}} \tag{26}$$

as symplectic manifolds. Note the symplectic actions of the loop group LG and of the Cartan subgroup T on $\mathcal{M}_{_{LG}}$ given, respectively, by $(h,\tau)\mapsto(h'h,\tau)$ and $(h,\tau)\mapsto(ht^{-1},\tau)$.

One may introduce the Darboux coordinates on the (complex version of) the model space $\mathcal{M}_{_{LG}}$ using the Gauss decomposition of the fields $h_{_L}$. For the $SU(2)$ group the decomposition is

$$h_{_L} \;=\; \begin{pmatrix}1 & \beta(x)\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ w(x) & 1\end{pmatrix}\begin{pmatrix}\psi(x) & 0\\ 0 & \psi(x)^{-1}\end{pmatrix}, \tag{27}$$

see Sect. 5 of [15]. Defining the modes

$$\begin{aligned}&\beta(x) \equiv \sum_n \beta_n\,\mathrm{e}^{-inx}\,, \qquad \gamma(x) \equiv \sum_n \gamma_n\,\mathrm{e}^{-inx} \;=\; ik\,\psi^{-2}(x)\,\partial(\psi^2(x)w(x))\,,\\
&\phi(x) \equiv \phi_{_0} + a_{_0}x + i\sum_{n\neq 0}\tfrac{1}{n}\,a_n\,\mathrm{e}^{-inx} \;=\; 2i\zeta\,\ln\psi(x)\end{aligned} \tag{28}$$

for $\zeta = \sqrt{\frac{k}{2}}$, one obtains the canonical Poisson brackets

$$\{a_n\,,\,a_m\} \;=\; -in\,\delta_{n,-m}\,, \qquad \{\phi_{_0}\,,\,a_{_0}\} \;=\; 1\,, \qquad \{\beta_n\,,\,\gamma_n\} \;=\; -i\,\delta_{n,-m} \tag{29}$$

with the other brackets vanishing. In terms of ϕ, β and γ,

$$J_{_L} \;=\; \begin{pmatrix} -\zeta\,\partial\phi-\beta\gamma & -ik\,\partial\beta+2\zeta\,\beta\partial\phi+\beta^2\gamma \\ -\gamma & \zeta\,\partial\phi+\beta\gamma \end{pmatrix}, \tag{30}$$

$$h_{_L} \;=\; \begin{pmatrix} (\Pi-\Pi^{-1})\psi+\beta\,\psi^{-1}Q & \beta\,\psi^{-1} \\ \psi^{-1}Q & \psi^{-1} \end{pmatrix}\begin{pmatrix} (\Pi-\Pi^{-1})^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \tag{31}$$

where $\Pi = \mathrm{e}^{-\pi i\,\zeta^{-1}a_0}$ is the monodromy of $\psi(x)$ and

$$Q(x) \;=\; \frac{1}{ik\,\Pi}\int\limits_x^{x+2\pi} \gamma(y)\,\psi(y)^2\,dy \tag{32}$$

is the "screening charge" appearing when solving for w in terms of γ: $w=\psi^{-2}Q/(\Pi-\Pi^{-1})$.

Whereas the 2-form Ω^{LG} involves the loop group geometry, the 2-form Ω^{PL}, with ρ given by Eq. (22), is related to the Poisson-Lie geometry. More precisely, Ω^{PL} defines a symplectic form on the space $\mathcal{M}^{PL}_{_G}=G\times\mathcal{A}_{_W}$ of pairs[4] $(g_{_0},\tau)$. For fixed τ, it determines a symplectic form Ω^{PL}_τ on the conjugacy class $\mathcal{C}_{_\tau}\subset G$ composed of the elements $g_{_0}\,\mathrm{e}^{2\pi i\tau}g_{_0}^{-1}\in G$. The conjugacy classes with the symplectic form Ω^{PL}_τ may be identified with the symplectic leaves of the Poisson-Lie group $G^*=\{(\gamma_+,\gamma_-)\in B_+\times B_-\}$ dual to the group G equipped with the Poisson-Lie structure induced by the standard r-matrix in $\mathbf{g}\otimes\mathbf{g}$ [34], see [15] for a short account. The identification is done via the map $(\gamma_+,\gamma_-)\mapsto\gamma_-\gamma_+^{-1}$. The conjugacy classes are, of course, the orbits of the adjoint action of G. More generally, there is a Poisson-Lie action of G on $\mathcal{M}^{PL}$ defined by $(g_{_0},\tau)\mapsto(gg_{_0},\tau)$. Note also a symplectic action of the Cartan subgroup on $\mathcal{M}^{PL}_{_G}$ given by $(g_{_0},\tau)\mapsto(g_{_0}t^{-1},\tau)$.

The symplectic leaves of G^* play in the Poisson-Lie category a role similar to that of the coadjoint orbits in the Lie category. The space $\mathcal{M}^{PL}_{_G}$ with the symplectic form Ω^{PL} may be interpreted as the model space of the Poisson-Lie group G, containing once each symplectic leaf of G^*. The choice (22) and the vertex-IRF parametrization (23) unravel this way a hidden Poisson-Lie symmetry of the chiral components of the WZW theory [14, 4, 23, 5]. In particular, we may express the chiral phase space $\widetilde{\mathcal{P}}_{_L}$ as the symplectic reduction (denoted by //) of the product of the loop group and the Poisson-Lie model spaces by the diagonal action of the Cartan subgroup T. The reduction imposes the constraint equating the τ components in both spaces and takes the orbit space of T:

$$\widetilde{\mathcal{P}}_{_L} \;\cong\; (\mathcal{M}_{_{LG}}\times\mathcal{M}^{PL}_{_G})\Big/\!\!\Big/T \;=\; (\mathcal{M}_{_{LG}}\times_{\mathcal{A}_W}\mathcal{M}^{PL}_{_G})\Big/T\,. \tag{33}$$

The representations (26) and (33) lend themselves easily to the (geometric) quantization.

First, the coadjoint orbits of $\widehat{LG}$ which pass through $\tau=\lambda/k\in\mathcal{A}_{_W}$, where k is a positive integer and λ is a weight, may be quantized by the Kirillov-Kostant method [30] (for fixed k, there is a finite number of such orbits). Upon quantization, they give rise to the irreducible highest weight representations of level k of the Kac-Moody algebra $\widehat{\mathbf{g}}$ ($\simeq$ the Lie algebra of $\widehat{LG}$) which act in the (infinite-dimensional) vector spaces $\mathcal{V}_{_{k,\lambda}}$. Quantization

[4]More precisely, one should consider the space of quadruples $(g_{_0},\tau,\gamma_\pm)$ s. t. $g_{_0}\,\mathrm{e}^{2\pi i\tau}g_{_0}^{-1}=\gamma_-\gamma_+^{-1}$.

of the chiral phase space $\mathcal{P}_L \cong \mathcal{M}_{LG}$ composed of the chiral fields g_L with monodromies of the form $\mathrm{e}^{2\pi i\tau}$ is then straightforward and gives the space of quantum states

$$\mathcal{H}_L \;=\; \mathop{\oplus}_{\lambda} \mathcal{V}_{k,\lambda}\,. \tag{34}$$

The Kac-Moody algebra action in the representation spaces and the Virasoro algebra one, induced from the latter by the Sugawara construction, quantize the infinitesimal versions of the classical LG and $Diff_+(S^1)$ symmetries of the chiral phase space. The space of states of the complete (left-right) quantum WZW theory is

$$\mathcal{H} \;=\; \mathop{\oplus}_{\lambda}\, \mathcal{V}_{k,\lambda} \otimes \overline{\mathcal{V}_{k,\lambda}}\,. \tag{35}$$

This mimics the diagonal way in which the classical phase space $\mathcal{P}$ is built from the coadjoint orbits of $\widehat{LG}$ in $\mathcal{P}_L$ and in $\mathcal{P}_R$. The overbar stands for the complex conjugation taking into account the opposite symplectic structure of the right-handed component of the phase space.

In a similar way, the symplectic leaves of G^* isomorphic to the conjugacy classes $\mathcal{C}_\tau$ with $\tau = \lambda/k$ may be quantized to the irreducible highest-weight representations of the quantum deformation $\mathcal{U}_q(\mathbf{g})$ of the enveloping algebra of $\mathbf{g}$ with the deformation parameter $q = \mathrm{e}^{\pi i/(k+h^\vee)}$ ($h^\vee$ denotes the dual Coxeter number of the Lie algebra $\mathbf{g}$). They act in the finite-dimensional spaces $\mathcal{V}_{q,\lambda}$. Quantization of the extended chiral phase space $\widetilde{\mathcal{P}}_L$ gives then rise to the space of states

$$\widetilde{\mathcal{H}}_L \;=\; \mathop{\oplus}_{\lambda}\, \mathcal{V}_{k,\lambda} \otimes \mathcal{V}_{q,\lambda}\,, \tag{36}$$

which is the quantum counterpart of the classical decomposition (33). As explained in [15] for $G = SU(2)$, one may quantize the chiral fields $g_L(x)$ (with general monodromies) so that, in the decomposition (23), $h(x)$ becomes a matrix of the "vertex operators" of the Kac-Moody algebra [35] and g_0^{-1} becomes its quantum-group counterpart. It should be stressed that in the theory without boundary, the quantum group degrees of freedom are superfluous and serve only to elucidate the chiral structure of the model. Below, we shall recover a similar coupling of the loop group and the quantum group degrees of freedom in the boundary WZW theory. In that case, however, both the loop group and the quantum group will describe physical degrees of freedom.

3 Phase space of the boundary WZW model

Let us consider now the WZW theory on the space-time M in the form of the strip $\mathbf{R}\times[0,\pi]$, see Fig. 2.

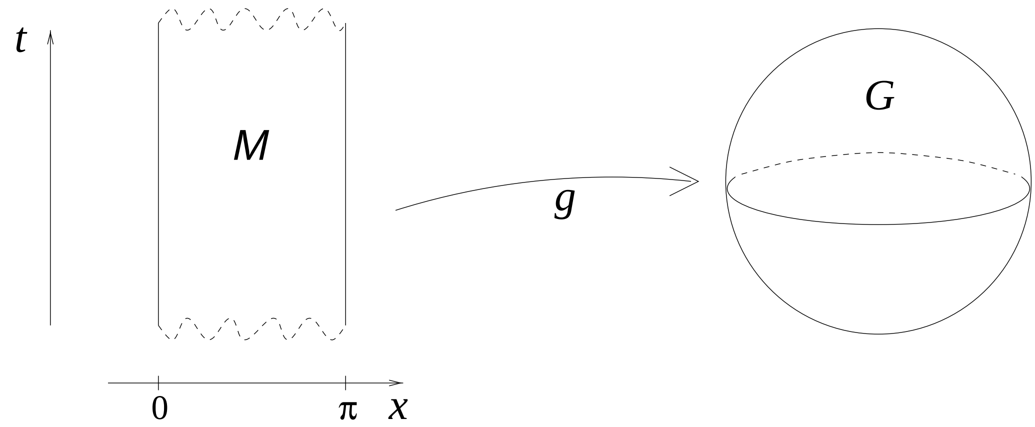

Fig. 2

Following [3], we shall impose on the fields $g: M \to G$ the boundary conditions requiring them to belong to fixed conjugacy classes on the components of the boundary:

$$g(t,0) \;\in\; \mathcal{C}_{\mu_0}\,, \qquad g(t,\pi) \;\in\; \mathcal{C}_{\mu_\pi}\,. \tag{37}$$

As before, $\mathcal{C}_\mu \;=\; \{\, g_0\, \mathrm{e}^{2\pi i \mu} g_0^{-1} \,|\, g_0 \;\in\; G\}$ and we shall take μ in $\mathcal{A}_W \;\subset\; \mathbf{t}$. This labels the conjugacy classes in a one-to-one way. The boundary conditions (37) generalize the Dirichlet conditions used for the abelian groups G. They will permit to preserve in the case with boundary the infinite dimensional LG and $Diff_+S^1$ symmetries (there are other choices of boundary conditions with the same effect).

The action of the model will be again given by Eq. (1) or, in the first order formalism, by Eq. (4). Now, however, a boundary term appears in the variation of the action:

$$\delta S(\Phi) \;=\; \int\limits_M \Phi^*(\iota_{\delta\Phi}\, d\alpha) \;+\; \int\limits_{\partial M} \Phi^*(\iota_{\delta\Phi}\, \alpha)\,. \tag{38}$$

The equation $\delta S(\Phi) = 0$ implies then, besides the bulk relations (7), also the boundary ones which require that

$$\Phi^*(\iota_X\, \alpha) \;=\; 0 \qquad \text{along} \qquad \partial M \tag{39}$$

for the vector fields X on P tangent to the boundary condition surface

$$\{(t,0\} \times \mathcal{C}_{\mu_0} \times \mathbf{g}^2 \;\cup\; \{(t,\pi)\} \times \mathcal{C}_{\mu_\pi} \times \mathbf{g}^2 \quad\subset\quad P\,.$$

Again, the first order variational equations are equivalent to Eqs. (5) supplemented by the relations $\delta S(g) = 0$ for the second order action. The latter, besides the bulk equation (8), require that

$$\mathrm{tr}\,(g^{-1}\delta g)\,(g^{-1}\partial_+ g)\, dx^+ \;-\; \mathrm{tr}\,(g^{-1}\delta g)\,(g^{-1}\partial_- g)\, dx^- \;=\; g^*(\iota_{\delta g}\,\omega) \tag{40}$$

on the vectors tangent to ∂M. Note that now the choice of the form ω enters the classical equations.

Let us choose the 2-form ω so that its restrictions ω_{μ} to the boundary conjugacy classes $\mathcal{C}_{\mu}$, where $\mu = \mu_0$ or $\mu = \mu_\pi$, take the form

$$\omega_{\mu}(\gamma) \;=\; \mathrm{tr}\,(h_0^{-1} dh_0)\, \mathrm{e}^{2\pi i \mu}\, (h_0^{-1} dh_0)\, \mathrm{e}^{-2\pi i \mu} \tag{41}$$

in the parametrization $\gamma = h_0\, \mathrm{e}^{2\pi i \mu}\, h_0^{-1}$ of $\mathcal{C}_{\mu}$. Equivalently,

$$\omega_{\mu}(\gamma) \;=\; \mathrm{tr}\,(\gamma^{-1} d\gamma)\,(1 - Ad_{\gamma})^{-1}(\gamma^{-1} d\gamma) \;=\; \tfrac{1}{2}\,\mathrm{tr}\,(\gamma^{-1} d\gamma)\, \tfrac{1+Ad_g}{1-Ad_g}\, (\gamma^{-1} d\gamma) \tag{42}$$

(the linear map $(1 - Ad_{\gamma})$ may be inverted on $\gamma^{-1}\delta\gamma$ if $\delta\gamma$ is tangent to the conjugacy class of γ). It is easy to check that $d\omega_{\mu}$ coincides with the restriction of the 3-form θ to $\mathcal{C}_{\mu}$ so that the choice (41) is consistent with the relation (2). For such a choice, the boundary equations (40) reduce to the relation

$$J_L \;=\; -J_R \qquad \text{on} \qquad \partial M \tag{43}$$

where, as before, $J_L = ik\, g\partial_+ g^{-1}$ and $J_R = ik\, g^{-1}\partial_- g$. Eqs. (43) are the starting point of the usual approach to the boundary WZW theory [29, 3]. We preferred, however, to start from the conditions (37) and the action functional because this will allow to determine the canonical structure of the boundary WZW model by following a well defined procedure. This procedure generalizes the approach sketched in the previous section to the case of space-times with boundary, see below.

In terms of the decomposition (9) of the classical solutions into the chiral components, still implied by the bulk equation (8), the boundary equation (43) is easily seen to be equivalent to the conditions

$$g_L(y + 2\pi) \;=\; g_L(y)\,\gamma \qquad \text{and} \qquad g_R(y) \;=\; g_L(-y)\, h_0^{-1} \tag{44}$$

which require that the chiral components be twisted-periodic and linked to each other. Note that, by themselves, these relations assure that the solution given by Eq. (9) takes values in fixed conjugacy classes on the boundary since they imply that

$$g(t, 0) \;=\; g_L(t)\, h_0\, g_L(t)^{-1} \qquad \text{and} \qquad g(t, \pi) \;=\; g_L(t - \pi)\,\gamma\, h_0\, g_L(t - \pi)^{-1}. \tag{45}$$

The boundary conditions (37) determine these conjugacy classes. We infer that

$$h_0 \;\in\; \mathcal{C}_{\mu_0} \qquad \text{and} \qquad h_\pi \equiv \gamma\, h_0 \;\in\; \mathcal{C}_{\mu_\pi}\,. \tag{46}$$

Consequently, the classical solutions on the strip take the form

$$g(t, x) \;=\; g_L(t + x)\, h_0\, g_L(t - x)^{-1} \;=\; g_L(t + x - 2\pi)\, h_\pi\, g_L(t - x)^{-1}\,, \tag{47}$$

where h_0, h_π and the monodromy γ of g_L are constrained by the relations (46).

Let us denote by $\mathcal{P}_{\mu_0 \mu_\pi}$ the space of such classical solutions. It forms the phase space of the WZW theory on the strip. As for the case of the cylinder, $\mathcal{P}_{\mu_0 \mu_\pi}$ possesses the canonical symplectic structure given by the symplectic form

$$\Omega_{\mu_0 \mu_\pi}(\delta_1 \Phi,\, \delta_2 \Phi) \;=\; \int\limits_{M_t} \Phi^*(\iota_{\delta_2 \Phi}\, \iota_{\delta_1 \Phi}\, d\alpha) \;-\; \int\limits_{\partial M_t} \Phi^*(\iota_{\delta_2 \Phi}\, \iota_{\delta_1 \Phi}\, \alpha)\,. \tag{48}$$

The boundary term is necessary to render the right hand side t-independent. Explicitly,

$$\Omega_{\mu_0\mu_\pi} \;=\; \tfrac{k}{4\pi}\Big\{\int_0^\pi \mathrm{tr}\,[-\delta(g^{-1}\partial_t g)\,g^{-1}\delta g \,+\, 2\,(g^{-1}\partial_+ g)\,(g^{-1}\delta g)^2]\,dx \\ +\,\omega_{\mu_0}(g(t,0)) \,-\, \omega_{\mu_\pi}(g(t,\pi))\Big\} \tag{49}$$

and its closedness is guaranteed by Eq. (2).

The loop group elements $h \in LG$ act naturally on the space of fields g which satisfy the boundary conditions (37) by

$$g(t,x) \;\longmapsto\; h(x^+)\; g(t,x)\; h(-x^-)^{-1}\,. \tag{50}$$

It is easy to see (for example, from the general form (47) of the solutions) that they map classical solutions to classical solutions. The resulting action of LG on $\mathcal{P}_{\mu_0\mu_\pi}$ preserves the symplectic structure. In fact the choice (41) is imposed by requiring these properties of the action (50). On the infinitesimal level the LG-action is generated by the current

$$J(t,x) \;=\; \begin{cases} \;\;J_L(t,x) & \text{for}\;\; 0\leq x\leq\pi\,, \\ -J_R(t,2\pi-x) & \text{for}\;\; \pi\leq x\leq 2\pi \end{cases} \tag{51}$$

which may be viewed as a periodic function of x^+ with period 2π. Similarly, the diffeomorphisms $D \in Diff_+S^1$ act on the space $\mathcal{P}_{\mu_0,\mu_\pi}$ by

$$g(t,x) \;\longmapsto\; g_L(D^{-1}(t+x))\; h_0\; g_L(D^{-1}(t-x))^{-1} \tag{52}$$

if g is given by Eq. (47). The action preserves the symplectic form. It is generated infinitesimally by the energy-momentum tensor $T(t,x) = \frac{1}{2k}\,\mathrm{tr}\,J(t,x)^2$, again a periodic function of x^+ with period 2π. As we see, the WZW theory on the strip defined as above conserves half of the infinite-dimensional symmetries of the theory on the cylinder.

In terms of the field g_L that parametrizes the classical solutions via Eq. (47), the symplectic form (49) becomes

$$\Omega_{\mu_0\mu_\pi} \;=\; \tfrac{k}{4\pi}\Big[\int_0^{2\pi} \mathrm{tr}\;(g_L^{-1}\delta g_L)\;\partial_x(g_L^{-1}\delta g_L)\,dx \;+\; \mathrm{tr}\;(g_L^{-1}\delta g_L)(0)\,(\delta\gamma)\,\gamma^{-1} \\ +\,\mathrm{tr}\;(\delta h_0)h_0^{-1}\,(\gamma^{-1}\delta\gamma) \;+\; \omega_{\mu_0}(h_0) \;-\; \omega_{\mu_\pi}(\gamma\, h_0)\Big]\,. \tag{53}$$

Note a vague resemblance to the modified chiral symplectic form $\widetilde\Omega_L$ discussed in the preceding section. It is even more instructive to rewrite the form $\Omega_{\mu_0\mu_\pi}$ in terms of the vertex-IRF parametrization (23) of the twisted periodic field g_L which results in the decomposition

$$g(t,x) \;=\;=\; h_L(t+x-2\pi)\; U\; h_L(t-x)^{-1} \tag{54}$$

of the classical solutions on the strip with the boundary conditions (37), where

$$U \;=\; \mathrm{e}^{2\pi i\tau}\, g_0^{-1} h_0\, g_0 \;=\; g_0^{-1} h_\pi\, g_0\,, \tag{55}$$

combines the non-diagonal monodromy, see Eq. (47). Inserting the parametrization (23) to (53), we obtain:

$$\Omega_{\mu_0\mu_\pi} \;=\; \tfrac{k}{4\pi}\int_0^{2\pi} \mathrm{tr}\;[(h^{-1}\delta h)\,\partial_x(h^{-1}\delta h) \;+\; 2i\tau\,(h^{-1}\delta h)^2 \;-\; 2i\,(\delta\tau)(h^{-1}\delta h)]\,dx$$

$$+\ \tfrac{k}{4\pi}\Big\{\mathrm{tr}\,(h_{_0}^{-1}\delta h_{_0})(h_{_\pi}^{-1}\delta h_{_\pi})\ +\ \omega_{_{\mu_0}}(h_{_0})\ -\ \omega_{_{\mu_\pi}}(h_{_\pi})\ +\ \omega_{_\tau}(\gamma)\ +\ 4\pi i\ \mathrm{tr}(\delta\tau)(g_{_0}^{-1}\delta g_{_0})\Big\}$$

$$\equiv\ \Omega^{LG}\ +\ \Omega^{bd}\,. \tag{56}$$

where $\gamma = g_{_0}\,\mathrm{e}^{2\pi i\tau}g_{_0}^{-1} = h_{_\pi}h_{_0}^{-1}$. Above $\omega_{_\tau}(\gamma)$ is really a form depending on the pair $(g_{_0},\tau)$ rather than on γ since the latter, unlike $h_{_0}$ and $h_{_\pi}$, is not restricted to a single conjugacy class. We recognize the symplectic form Ω^{LG} of the loop group model space $\mathcal{M}_{_{LG}}$ as the first part of $\Omega_{_{\mu_0\mu_\pi}}$. The next section is devoted to the interpretation of the second part Ω^{bd} involving the boundary data (h_0, h_π, g_0, τ) defined modulo simultaneous adjoint action of G on $h_{_0}$ and $h_{_\pi}$ and left action on g_0.

4 Relation to the Chern-Simons theory

4.1 CS theory on a sphere

The 2-form Ω^{bd} may be identified with the symplectic structure on the phase space of the CS theory on $S^2\times\mathbf{R}$ with three Wilson lines. We shall represent S^2 as the complex projective plane $\mathbf{C}P^1\cong\mathbf{C}\cup\{\infty\}$ and we shall fix 3 punctures on it, say at points points 0, π and $w\equiv\frac{1}{2}\pi+2i$, see Figs. 3.

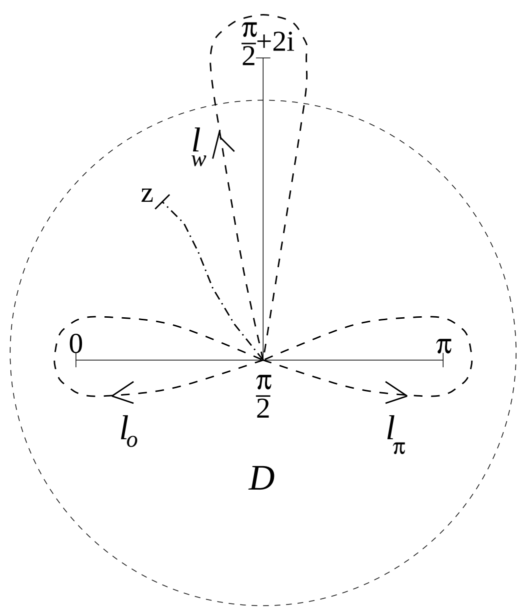

Fig. 3

Let us consider flat unitary gauge potentials (connections) A on $\mathbf{C}P^1\backslash\{0,\pi,w\}$ with values in the Lie algebra $\mathbf{g}$ and such that

$$A \;=\; \begin{cases} -\eta_0\,\mu_0\,\eta_0^{-1}\;\frac{1}{2i}(\frac{dz}{z}-\frac{d\bar z}{\bar z}) & \text{around}\quad 0\,,\\ \eta_\pi\mu_\pi\eta_\pi^{-1}\,\frac{1}{2i}(\frac{dz}{z-\pi}-\frac{d\bar z}{\bar z-\pi}) & \text{around}\quad \pi\,,\\ -\eta_w\,\tau\,\eta_w^{-1}\;\frac{1}{2i}(\frac{dz}{z-\frac{1}{2}\pi-2i}-\frac{d\bar z}{\bar z-\frac{1}{2}\pi+2i}) & \text{around}\quad w\,,\end{cases} \tag{57}$$

where $\eta_0,\,\eta_\pi,\,\eta_w\in G$ and where $\mu_0,\,\mu_\pi$ and τ are, as before, in the positive Weyl alcove $\mathcal{A}_W\subset\mathbf{t}$, the first two fixed and the last arbitrary. The closed 2-form on the infinite-dimensional space of flat gauge potentials A with the behavior (57) around the punctures,

$$\Omega^{CS} \;=\; -\tfrac{k}{4\pi}\int\limits_{\mathbf{C}}\mathrm{tr}\,(\delta A)^2\;+\;k i\,\mathrm{tr}\,[-\,\mu_0\,(\eta_0^{-1}\delta\eta_0)^2\;+\;\mu_\pi\,(\eta_\pi^{-1}\delta\eta_\pi)^2$$
$$-\,\tau\,(\eta_w^{-1}\delta\eta_w)^2\;+\;(\delta\tau)\,(\eta_w^{-1}\delta\eta_w\,]\,, \tag{58}$$

is invariant under the gauge transformations $h:\mathbf{C}P^1\to G$ constant around the punctures acting on the gauge potentials by

$$A\;\longmapsto\;h\,A\,h^{-1}\;-\;i\,(dh)h^{-1}\,. \tag{59}$$

It descends to the quotient space $\mathcal{P}^{CS}_{\mu_0\mu_\pi}$ making it a finite-dimensional symplectic manifold that may be identified as the phase space of the CS theory on $\mathbf{C}P^1\times\mathbf{R}$ with timelike Wilson lines passing through the punctures, see Fig. 4.

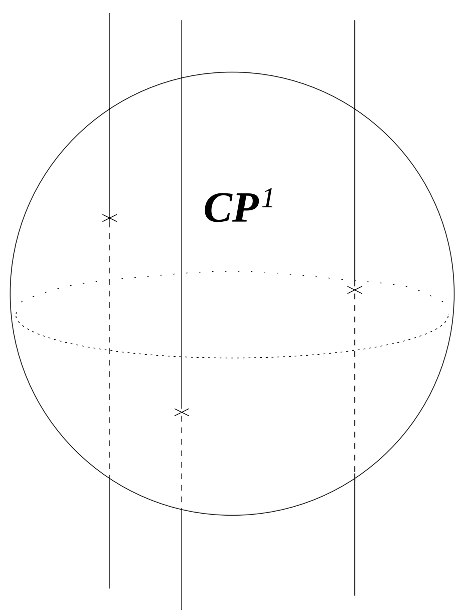

Fig. 4

We shall keep the notation Ω^{CS} for the symplectic form on $\mathcal{P}^{CS}_{\mu_0\mu_\pi}$. As before, the whole construction may be repeated in the complex setup where we relax unitarity of the connections and end up with a complex symplectic manifold.

The symplectic forms on the phase spaces of the CS theory on general punctured Riemann surfaces have been explicitly computed in terms of the holonomy of flat connections in the reference [2], see Theorem 1 therein. The idea of that computation is simple. One renders the surface simply connected by cutting it (in our case along the inverted letter T in Fig. 3).

On the cut surface, any flat connection A is pure gauge so that if one defines

$$g(z) \;=\; \overleftarrow{\mathrm{e}}^{\,i\int_{\ell_z} A} \tag{60}$$

for any path ℓ_z in the cut surface connecting the base point ($\frac{1}{2}\pi - 0 + i0$ in our case) to z, where $\overleftarrow{\mathrm{e}}$ denotes the path-ordered exponential, then $A = \frac{1}{i}(dg)g^{-1}$. The identity

$$\mathrm{tr}\;(\delta A)^2 \;=\; -d\,[\mathrm{tr}\,(g^{-1}\delta g)\,d(g^{-1}\delta g)] \tag{61}$$

permits to replace the surface integral in the definition (58) by the integral along the boundary of the cut surface which forms a contour that may be decomposed into the generators of the fundamental group. The rest of the argument is a straightforward, although tedious, bookkeeping.

For the case at hand with three punctures in the complex projective plane, the local behavior (57) around the punctures assures that the holonomy of A takes values in prescribed conjugacy classes:

$$h_0 \equiv \overleftarrow{\mathrm{e}}^{\,i\int_{\ell_0} A} \in \mathcal{C}_{\mu_0}\,, \quad h_\pi \equiv \overleftarrow{\mathrm{e}}^{\,i\int_{\ell_\pi} A} \in \mathcal{C}_{\mu_\pi}\,, \quad \gamma \equiv \overleftarrow{\mathrm{e}}^{\,i\int_{\ell_w} A} \in \mathcal{C}_{\mu_\tau}\,, \tag{62}$$

where h_0, h_π and $\gamma = h_\pi h_0^{-1}$ denote now the parallel transporters in the gauge potential A along the closed paths ℓ_0, ℓ_π and ℓ_w starting at $\frac{1}{2}\pi$, see Fig. 3. Writing $\gamma = g_0\,\mathrm{e}^{2\pi i\tau} g_0^{-1}$, one obtains the identification

$$\mathcal{P}^{CS}_{\mu_0\mu_\pi} \;\cong\; \Big\{ (h_0, h_\pi, g_0, \tau) \in \mathcal{C}_{\mu_0} \times \mathcal{C}_{\mu_\pi} \times G \times \mathcal{A}_W \;\Big|\; h_\pi h_0^{-1} = g_0\,\mathrm{e}^{2\pi i\tau} g_0^{-1} \Big\}\Big/ G\,. \tag{63}$$

The simultaneous adjoint action of G on h_0 and h_π and the left action on g_0, whose orbit space is taken above, is induced on the holonomy by the local gauge transformations of the gauge potentials A. It appears then that, expressed in the language of (h_0, h_π, g_0, τ), the symplectic form Ω^{CS} on $\mathcal{P}^{CS}_{\mu_0\mu_\pi}$, see (58), coincides with the Ω^{bd} part of the symplectic form $\Omega_{\mu_0\mu_\pi}$ given by Eq. (56) on the phase space of the boundary WZW model. Note the symplectic action of the Cartan subgroup T on $\mathcal{P}^{CS}_{\mu_0\mu_\pi}$ by $g_0 \mapsto g_0 t^{-1}$.

The phase space $\mathcal{P}_{\mu_0\mu_\pi}$ of the boundary WZW theory may be viewed as the symplectic reduction with respect to the diagonal action of the Cartan subgroup T of the product of the model space $\mathcal{M}_{LG}$ for the loop group and of $\mathcal{P}^{CS}_{\mu_0\mu_\pi}$:

$$\mathcal{P}_{\mu_0\mu_\pi} \;\cong\; (\mathcal{M}_{LG} \times \mathcal{P}^{CS}_{\mu_0\mu_\pi}) /\!\!/ T \;=\; (\mathcal{M}_{LG} \times_{\mathcal{A}_W} \mathcal{P}^{CS}_{\mu_0\mu_\pi}) \big/ T\,, \tag{64}$$

where the fiber product over $\mathcal{A}_W$ equates the τ components of $\mathcal{M}_{LG}$ and of $\mathcal{P}_{\mu_0\mu_\pi}$. This is the main structural result of thhis subsection, which should be compared with the preceding result (33) about the structure of the chiral phase space of the WZW theory on the cylinder.

As is well known, the phase space of the CS theory on a punctured Riemann surface, with the holonomies around the punctures constrained to the conjugacy classes $\mathcal{C}_{\lambda_i/k}$ where λ_i are weights, may be quantized. Upon quantization it gives rise to the finite-dimensional space $W_{k,(\lambda_i)}$ of the conformal blocks of the WZW theory with insertions of primary fields in representations of G with highest weights λ_i. Consequently, the classical decomposition (64) suggests the following realization of the quantum space of states of the WZW theory with the boundary conditions (37) if $\mu_0 = \lambda_0/k$ and $\mu_\pi = \lambda_\pi/k$ where λ_0 and λ_π are weights:

$$\mathcal{H}_{\lambda_0\lambda_\pi} \;=\; \mathop{\oplus}_{\lambda}\, \mathcal{V}_{k,\lambda} \otimes W_{k,\lambda_0\bar{\lambda}_\pi\lambda}\,, \tag{65}$$

where the sum is over the weights λ with λ/k in the positive Weyl alcove $\mathcal{A}_W$. By definition, $\bar{\mu}$ labels the conjugacy class inverse to $\mathcal{C}_\mu$ and $\bar{\mu}_\pi = \bar{\lambda}_\pi/k$. The replacement of μ_π by $\bar{\mu}_\pi$ is due to the opposite orientation of the contour ℓ_π in Fig 3.

The decomposition (65) is consistent with results of the general theory of conformal boundary conditions [9, 10]. That theory states that, for the so called diagonal models whose examples are provided by the WZW theories with simply connected groups, the boundary conditions are in a one-to-one correspondence with the primary fields of the bulk model. Indeed, in our case[5], both are labeled by the weights λ in $k\mathcal{A}_W \subset \mathbf{t}$. Moreover, the general theory asserts that the irreducible representations of the chiral algebra (in our case, of the Kac-Moody algebra) appear in the boundary theory Hilbert spaces with the multiplicities given by the (Verlinde) dimensions of the spaces of 3-point conformal blocks

$$N_{\lambda_0\lambda}^{\ \ \lambda_\pi} \;=\; \dim\, W_{k,\lambda_0\bar{\lambda}_\pi\lambda}\,, \tag{66}$$

in agreement with the decomposition (65). As we shall see below, our classical results allow, however, for more. They permit, for example, to quantize naturally the basic fields of the boundary WZW model.

4.2 CS theory on a disc

The result of the last subsection permits to establish an even more direct relation between the boundary WZW model and the CS theory on a 3-manifold with boundary. Let us consider the CS theory on $D\times\mathbf{R}$ where D is a disc of radius 1 centered at $\frac{1}{2}\pi$, see Fig. 3, with two timelike Wilson lines passing through the punctures at 0 and π. The phase space $\mathcal{P}^{CS}_{D,\mu_0\mu_\pi}$ of the theory is composed of flat connections A_D on D with the representation as in (57) around 0 and π, modulo gauge transformation constant around the punctures and equal to 1 on ∂D. The symplectic form Ω^{CS}_D is given by the first line of (58) with the integral restricted to D. The phase space $\mathcal{P}^{CS}_{D,\mu_0\mu_\pi}$ may be easily identified with the phase space $\mathcal{P}_{\mu_0\mu_\pi}$ of the boundary WZW model using the map

$$A_D \;\longmapsto\; (h_0, h_\pi, \tau,\, h)\,, \tag{67}$$

[5] See [24] for more details on how quantization chooses the discrete family of conjugacy classes.

where h_0 and h_π are defined as in (62) and describe the holonomy of A_D around the punctures, $h_\pi h_0^{-1} = g_0 \, \mathrm{e}^{2\pi i \tau} g_0^{-1}$, and

$$h(x) = g(\tfrac{1}{2}\pi + i\,\mathrm{e}^{i\,x})\, g_0\, \mathrm{e}^{-i\,\tau\, x}\,, \tag{68}$$

with $g(z)$ given by (60). It is easy to see that h is periodic, i.e. that it belongs to the loop group. The gauge transformations of A_D induce a simultaneous adjoint action of G on the holonomy h_0 and h_π and do not change h. The map of the orbits is 1 to 1 and establishes the isomorphism between $\mathcal{P}^{CS}_{D,\mu_0\mu_\pi}$ and $\mathcal{P}_{\mu_0\mu_\pi}$. It remains to identify the symplectic structures of two phase spaces. This may be done along the lines of [2] or using directly the result of that reference. In the latter case, we extend a connection A_D on D to a flat connection A on $\mathbf{C}P^1$ with three punctures, with the behavior (57) around them, and write

$$\Omega^{CS}_D \;=\; \Omega^{CS} \;+\; \frac{k}{4\pi} \int\limits_{\mathbf{C}\setminus D} \mathrm{tr}\;(\delta A)^2 \;+\; k i\,\mathrm{tr}\,[\tau(\eta_w^{-1}\delta\eta_w)^2 \;-\; (\delta\tau)(\eta_w^{-1}\delta\eta_w\,]\,, \tag{69}$$

see (58). As we have discussed, Ω^{CS} reproduces the boundary part Ω^{bd} of the symplectic form $\Omega_{\mu_0\mu_\pi}$ of $\mathcal{P}_{\mu_0\mu_\pi}$, see (56). It remains to see that the other term reproduces Ω^{LG}. This is an old result [13] which says that the CS phase spaces on a disc with one puncture may be identified with the coadjoint orbits of $\widehat{LG}$. It may be easily established using (61) and integrating by parts.

The relation of the the boundary WZW theory to the CS theory on twice punctured disc is certainly worth pursuing further. As mentioned in Introduction, it may lead to new applications of the boundary theory. It is also a source of natural structures in the boundary models that are less visible in the original formulation. It also raises a natural question about the interpretation of the CS theory on a disc with more than two punctures.

5 Relation to the Poisson-Lie groups

Reference [2] contains another valuable result, stated in Theorem 2 therein. It realizes the (complex versions) of the CS theory phase spaces in Poisson-Lie terms. Let us recall how this is done. Consider the product space $\mathcal{C}_{\mu_0} \times \mathcal{C}_{\bar\mu_\pi} \times \mathcal{M}^{PL}_G$ composed of the elements $(\gamma_0, \gamma_\pi, (\sigma_w, \tau))$ with

$$\begin{aligned} \gamma_0 &= \sigma_0 \mathrm{e}^{2\pi i \mu_0}\, \sigma_0^{-1} &&= \gamma_{0-}\gamma_{0+}^{-1}\,, \\ \gamma_\pi &= \sigma_\pi \mathrm{e}^{-2\pi i \mu_\pi}\, \sigma_\pi^{-1} &&= \gamma_{\pi-}\gamma_{\pi+}^{-1}\,, \\ \gamma_w &= \sigma_w \mathrm{e}^{2\pi i \tau}\, \sigma_w^{-1} &&= \gamma_{w-}\gamma_{w+}^{-1}\,. \end{aligned} \tag{70}$$

Recall that the Poisson-Lie model space $\mathcal{M}^{PL}_G$ comes equipped with the symplectic form Ω^{PL}, see Eq. (25). In turn, upon fixing τ, Ω^{PL} induces the symplectic forms Ω^{PL}_τ on the conjugacy classes $\mathcal{C}_\tau$ which are identified with the symplectic leaves of the dual Poisson-Lie group G^*. The product space $\mathcal{C}_{\mu_0} \times \mathcal{C}_{\bar\mu_\pi} \times \mathcal{M}^{PL}_G$ may be equipped with the symplectic structure

$$\Omega^{PL}_{\mu_0} \,+\, \Omega^{PL}_{\bar\mu_\pi} \,+\, \Omega^{PL}\,. \tag{71}$$

Define now the map

$$\mathcal{C}_{\mu_0}\times\mathcal{C}_{\bar\mu_\pi}\times\mathcal{M}^{PL}_{G}\ni(\gamma_0,\gamma_\pi,(\sigma_w,\tau))\ \mapsto\ (h_0,h_\pi,(g_0,\tau))\in\ \mathcal{C}_{\mu_0}\times\mathcal{C}_{\bar\mu_\pi}\times G\times\mathcal{A}_W \quad (72)$$

by setting

$$h_0\ =\ \gamma_0\,,\qquad h_\pi^{-1}\ =\ \gamma_{0+}\gamma_\pi\gamma_{0+}^{-1}\,,\qquad g_0\ =\ \gamma_{0+}\gamma_{\pi+}\sigma_w\,. \tag{73}$$

Note that if we set $\gamma=g_0\,\mathrm{e}^{2\pi i\tau}g_0^{-1}$ then $\gamma=\gamma_{0+}\gamma_{\pi+}\gamma_w\gamma_{\pi+}^{-1}\gamma_{0+}^{-1}$. The separate Poisson-Lie action of G on $\mathcal{C}_{\mu_0}$, $\mathcal{C}_{\bar\mu_\pi}$ and $\mathcal{M}^{PL}_G$ given by the adjoint action on γ_0 and γ_π and the left action on σ_w has a (twisted-)diagonal version. This version consists of the simultaneous adjoint action of G on h_0, and h_π and the left action on g_0 and may be used to perform a Poisson-Lie version of the symplectic reduction of the product manifold $\mathcal{C}_{\mu_0}\times\mathcal{C}_{\bar\mu_\pi}\times\mathcal{M}^{PL}_G$. In concrete terms, the reduction imposes the condition

$$\gamma_{0-}\gamma_{\pi-}\gamma_{w-}\ =\ 1\ =\ \gamma_{0+}\gamma_{\pi+}\gamma_{w+} \tag{74}$$

which is the same as $h_\pi h_0^{-1}=\gamma$ and passes to the space of orbits of the (twisted-)diagonal Poisson-Lie action of G. The symplectic form (71) descends to the reduced space. Recalling from the previous section the realization (63) of the phase space of the CS theory on the projective plane with three punctures, we infer that the map (73) induces the isomorphism[6]

$$\mathcal{P}^{CS}_{\mu_0\mu_\pi}\quad\cong\quad(\mathcal{C}_{\mu_0}\times\mathcal{C}_{\bar\mu_\pi}\times\mathcal{M}^{PL}_G)\,/\!\!/\,G\,. \tag{75}$$

A direct calculation [2] shows then that this is an isomorphism of symplectic manifolds.

Note that in terms of the parametrization (73), the monodromy part (55) in the decomposition (54) of the classical solutions of the boundary theory take the form

$$U\ =\ \sigma_w^{-1}\gamma_{\pi-}^{-1}\gamma_{\pi+}\sigma_w\,. \tag{76}$$

Below, we shall quantize these expressions. This will permit an explicit construction of the action of the quantum bulk fields $g(t,x)$ in the spaces of states of the boundary WZW theory.

6 Quantization of the boundary theory

6.1 The space of states

The isomorphism (75) has its counterpart at the quantum level which allows for another presentation of the space of states of the boundary theory, see (65). Under quantization, the symplectic space $\mathcal{C}_{\mu_0}\times\mathcal{C}_{\mu_\pi}\times\mathcal{M}^{PL}_G$ for $\mu_{0,\pi}=\lambda_{0,\pi}/k$ becomes $\bigoplus_\lambda\mathcal{V}_{q,\lambda_0\bar\lambda_\pi\lambda}$ in the shorthand notation $\mathcal{V}_{q,(\lambda_i)}\equiv\otimes_i\mathcal{V}_{q,\lambda_i}$ for the tensor product of the highest weight representations of the deformed enveloping algebra $\mathcal{U}_q(\mathbf{g})$. The diagonal Poisson-Lie action of G gives rise on the quantum level to the diagonal (coproduct induced) action of $\mathcal{U}_q(\mathbf{g})$ in the latter space.

[6] More exactly, an isomorphism between open dense subspaces.

In the first approximation, the subspace $\underset{\lambda}{\oplus}\ \mathcal{V}^{inv}_{q,\lambda_0\bar\lambda_\pi\lambda}$ of the invariant tensors of that action gives the space of states corresponding to the symplectic reduction $(\mathcal{C}_{\mu_0}\times\mathcal{C}_{\bar\mu_\pi}\times\mathcal{M}^{PL}_G)\,/\!\!/\,G$. More precisely, the subspaces of invariants $\mathcal{V}^{inv}_{q,(\lambda_i)}\subset\mathcal{V}_{q,(\lambda_i)}$ may be equipped with a semi-positive scalar product (coming from natural hermitian forms on the spaces $\mathcal{V}_{q,\lambda}$) and one should divide by the subspaces of null-vectors. The quotient spaces $\mathcal{W}_{q,(\lambda_i)}$ are isomorphic to the spaces $\mathcal{W}_{k,\lambda_0\bar\lambda_\pi\lambda}$ of the conformal blocks of the WZW theory. Consequently, we obtain the following presentation of the space of states of the boundary WZW theory defined in Eq. (65):

$$\mathcal{H}_{\lambda_0\lambda_\pi}\ \cong\ \underset{\lambda}{\oplus}\ \mathcal{V}_{k,\lambda}\otimes\mathcal{W}_{q,\lambda_0\bar\lambda_\pi\lambda}\ \subset\ \mathcal{H}_L\otimes\mathcal{H}^{bd}_{\lambda_0\lambda_\pi}\,, \tag{77}$$

where the chiral space of states $\mathcal{H}_L$ is given by Eq. (34) and

$$\mathcal{H}^{bd}_{\lambda_0\lambda_\pi}\ =\ \underset{\lambda}{\oplus}\ \mathcal{W}_{q,\lambda_0\bar\lambda_\pi\lambda}\,. \tag{78}$$

This realization of the space $\mathcal{H}_{\lambda_0\lambda_\pi}$ will allow to define the action of the quantized bulk fields $g(t,x)$ in $\mathcal{H}_{\lambda_0\lambda_\pi}$ by finding the quantum version of the decomposition (54) of the classical field. As in the bulk case, quantization of the factors h_L and h_L^{-1} will be given by the vertex operators of the Kac-Moody algebra whereas the monodromy factor U will be realized by operators acting in the space $\mathcal{H}^{bd}_{\lambda_0\lambda_\pi}$. In what follows, we shall carry out this construction in detail for the case of the group $SU(2)$.

6.2 The Kac-Moody vertex operators

For $G=SU(2)$, the weights λ such that $\lambda/k\in\mathcal{A}_W$ are labeled by spins $j=0,\frac{1}{2},\dots,\frac{k}{2}$: $\lambda=\begin{pmatrix} j & 0\\ 0 & -j\end{pmatrix}$. It was shown in [7] that one may realize the spaces $\mathcal{V}_{k,j}$ carrying the irreducible highest weight representations of the Kac-Moody algebra $\widehat{su}(2)$ as the cohomology of a complex of Fock spaces $\mathcal{F}_\alpha$. The latter carry representations of the CCR algebra[7]

$$[a_n,a_m]=n\,\delta_{n,-m}\qquad[\beta_n,\gamma_m]=\delta_{n,-m} \tag{79}$$

with all the other commutators vanishing. $\mathcal{F}_\alpha$ are built by applying the creation operators a_n, β_{n+1} and γ_n with $n<0$ to the vacuum vector $|\alpha\rangle$ s.t. $a_0\,|\alpha\rangle=\alpha|\alpha\rangle$ and $a_n|\alpha\rangle=\beta_n|\alpha\rangle=\gamma_{n-1}|\alpha\rangle=0$ for $n>0$. One introduces the free field vertex operators depending on the real variable x, the Wick ordered exponentials of the (chiral) free field $\phi(x)=\phi_0+a_0x+i\sum\limits_{n\neq0}\frac{1}{n}\,a_n\,\mathrm{e}^{-inx}$,

$$\psi_\eta(x)\ =:\mathrm{e}^{-i\frac{\eta}{2\xi}\phi(x)}:\ \equiv\ \mathrm{e}^{-i\frac{\eta}{2\xi}\phi_0}\,\mathrm{e}^{-i\frac{\eta}{2\xi}a_0\,x}\,\mathrm{e}^{\frac{\eta}{2\xi}\sum\limits_{n<0}\frac{1}{n}a_n\mathrm{e}^{-inx}}\ \mathrm{e}^{\frac{\eta}{2\xi}\sum\limits_{n>0}\frac{1}{n}a_n\mathrm{e}^{-inx}}\,. \tag{80}$$

Above, $\xi=\sqrt{\frac{k+2}{2}}$ and $\mathrm{e}^{-i\frac{\eta}{2\xi}\phi_0}|\alpha\rangle=|\alpha-\frac{\eta}{2\xi}\rangle$ and it commutes with all the generators of the CCR algebra but a_0. The operators $\psi_\eta(x)$ are twisted-periodic: $\psi_\eta(x+2\pi)=\psi_\eta(x)\,\mathrm{e}^{-\frac{\pi i}{\xi}\eta a_0}$ and they satisfies the commutation relations

$$\psi_\eta(x)\ \psi_{\eta'}(x')\ =\ \mathrm{e}^{\frac{\pi i\eta\eta'}{k+2}\,[E(\frac{x-x'}{2\pi})+\frac{1}{2}]}\ \psi_{\eta'}(x')\ \psi_\eta(x)\,, \tag{81}$$

[7] In [7], β_n and γ_n are denoted, respectively, ω_n and $\omega_n^\dagger$.

where $E(\cdot)$ denotes the "entier" function. Introduce also the β, γ fields $\beta(x) = \sum \beta_n \, \mathrm{e}^{-inx}$, $\gamma(x) = \sum \gamma_n \, \mathrm{e}^{-inx}$ which are periodic in x and which satisfy the commutation relation

$$[\beta(x),\, \gamma(x')] \;=\; 2\pi\, \delta(x-x')\,. \tag{82}$$

The quantum version of the free field construction (30) of the current reads

$$J \;\equiv\; \begin{pmatrix} J^3 & J^+ \\ J^- & -J^3 \end{pmatrix} \;=\; \begin{pmatrix} -\xi\,\partial\phi - :\beta\gamma: & -ik\,\partial\beta \,+\, 2\xi\,\beta\,\partial\phi + :\beta^2\gamma: \\ -\gamma & \xi\,\partial\phi + :\beta\gamma: \end{pmatrix}. \tag{83}$$

It goes back to Wakimoto [36] and may be easily rewritten in terms of the current modes such that $J(x) = \sum J_n \, \mathrm{e}^{-inx}$. The quantized current satisfies for $|x - x'| < 2\pi$ the commutation relations

$$[J(x)_{_1},\, J(x')_{_2}] \;=\; 2\pi\,\delta(x-x')\,[J(x)_{_1},\, C_{_{12}}] \;+\; 2\pi i k\,\delta'(x-x')\,C_{_{12}}\,, \tag{84}$$

a quantum counterpart of the Poisson bracket of the classical current. The action of the quantized current turns the Fock spaces $\mathcal{F}_\alpha$ into the modules of the $\widehat{su}(2)$ affine Kac-Moody algebra. The (screening) operators

$$Q(x) = \tfrac{1}{ik}\, \mathrm{e}^{\frac{\pi i}{\xi}a_0} \int\limits_x^{x+2\pi} \gamma(y)\,\psi_{_2}(y)\,dy \tag{85}$$

are nilpotent $Q(x)^{k+2} = 0$ and define for $j = 0, \frac{1}{2}, \ldots, \frac{k}{2}$ and $Q \equiv Q(x)$ an x-independent complex

$$\ldots \;\stackrel{Q^{2j+1}}{\longrightarrow}\; \mathcal{F}_{\frac{k+1-j}{\xi}} \;\stackrel{Q^{k+1-2j}}{\longrightarrow}\; \mathcal{F}_{\frac{j}{\xi}} \;\stackrel{Q^{2j+1}}{\longrightarrow}\; \mathcal{F}_{-\frac{j+1}{\xi}} \;\stackrel{Q^{k+1-2j}}{\longrightarrow}\; \ldots \tag{86}$$

of $\widehat{su}(2)$-modules whose middle cohomology gives the irreducible highest weight module $\mathcal{V}_{_{k,j}}$ of $\widehat{su}(2)$, see [7]. The energy-momentum tensor

$$T \;=\; \tfrac{1}{2} :(\partial\phi)^2: - \tfrac{1}{2\xi}\,\partial^2\phi \,-\, :\gamma\,\partial\beta: \tag{87}$$

satisfies the Virasoro commutation relations

$$\begin{aligned}[T(x),\, T(x')] \;=&\; -4\pi i\,\delta'(x-x')\,T(x') \,+\, 2\pi i\,\delta(x-x')\,\partial T(x') \\ &- \tfrac{\pi i k}{2(k+2)}\,(\delta'(x-x') + \delta'''(x-x'))\end{aligned} \tag{88}$$

corresponding to the value $\frac{3k}{k+2}$ of the Virasoro central charge.

Quantization of the $SU(2)$-valued fields $h_{_L}(x)$ may be guessed from the classical free field representation (31). In fact, the last term, the matrix involving $(\Pi - \Pi^{-1})^{-1}$, has to be handled with care to avoid singularities. We find it convenient to reshuffle such terms between the current algebra and the quantum group degrees of freedom and to introduce four different matrices of quantum operators, $\chi(x)$, $\widetilde{\chi}(x)$, $u(x)$ and $\widetilde{u}(x)$. The first one,

$$\chi \;=\; \begin{pmatrix} \beta\,\psi_{_{-1}}\,Q & \beta\,\psi_{_{-1}} \\ \psi_{_{-1}}\,Q & \psi_{_{-1}} \end{pmatrix}. \tag{89}$$

should be thought of as quantization of the field h'_{L}, a modified version of h_{L} with the factor involving $(\Pi-\Pi^{-1})^{-1}$ on the right hand side of (31) dropped. Note that, compared to the classical expression (31), we have also dropped the term involving $\psi(x)$. This is a more delicate Wick-ordering renormalization effect. The components of $\chi(x)$ form the Kac-Moody vertex operators that descend to the Fock-space cohomology, see [7]. As the result, they may be viewed as operators acting in the chiral space of states[8]

$$\mathcal{H}_{L} \;=\; \mathop{\oplus}_{j=\frac{1}{2},\dots,\frac{k}{2}} \mathcal{V}_{k,j}\,. \tag{90}$$

The components χ_{a1} lower the value of j by $\frac{1}{2}$ and the χ_{a2} ones raise it by $\frac{1}{2}$.

The other matrices of operators that we shall consider are modified versions of χ. They are defined as follows:

$$\widetilde{\chi} \;=\; \begin{pmatrix} \psi_{-1} & -\beta\,\psi_{-1} \\ -\psi_{-1}\,Q & \beta\,\psi_{-1}\,Q \end{pmatrix}, \qquad\quad u \;=\; \chi\,\frac{1}{[p]}\,, \qquad\quad \widetilde{u} \;=\; \widetilde{\chi}\,\frac{1}{[p]}\,, \tag{91}$$

where $[p]$ stands for the q-deformation of $p\equiv 2j+1$: $[p]\equiv\frac{q^p-q^{-p}}{q-q^{-1}}$ with $q\equiv \mathrm{e}^{-\frac{\pi i}{k+2}}$. The field u will play the role of quantization of $h''_{L}=h'_{L}(\Pi-\Pi^{-1})^{-1}$, $\widetilde{u}$ of quantization of ${h'_{L}}^{-1}$ and $\widetilde{\chi}$ as that of ${h''_{L}}^{-1}$. On $\mathcal{H}_{L}$ we have the following commutation relations with the energy-momentum tensor

$$[T(x),\,\chi(x')] \;=\; -\tfrac{3\pi i}{2(k+2)}\,\delta'(x-x')\,\chi(x') \;+\; 2\pi i\,\delta(x-x')\,\partial\chi(x') \tag{92}$$

and similarly for $\widetilde{\chi}$, u and $\widetilde{u}$. They mean that all these fields are primary with conformal weight $\Delta_{1/2}=\frac{3k}{4(k+2)}$ for the Virasoro action induced by T on the cohomology of the Fock spaces. We also have the commutation relations with the current

$$[J(x)_{1},\,\chi(x')_{2}] \;=\; 2\pi\,\delta(x-x')\,C_{12}\,\chi(x')_{2} \tag{93}$$

and the same for u and

$$[J(x)_{1},\,\widetilde{\chi}(x')_{2}] \;=\; -2\pi\,\delta(x-x')\,\widetilde{\chi}(x')_{2}\,C_{12} \tag{94}$$

and the same for $\widetilde{u}$, all for $|x-x'|<2\pi$. They express the fact that the fields form primary spin $\frac{1}{2}$ multiplets of the current algebra. Finally, we also have on $\mathcal{H}_{L}$ the exchange relations

$$\chi(x)_{1}\,\chi(x')_{2} \;=\; \chi(x')_{2}\,\chi(x)_{1}\;D^{\pm}\,, \qquad\quad \widetilde{\chi}(x)_{1}\,\widetilde{\chi}(x')_{2} \;=\; \widetilde{D}^{\pm}\;\widetilde{\chi}(x')_{2}\,\widetilde{\chi}(x)_{1}\,, \tag{95}$$

$$u(x)_{1}\,u(x')_{2} \;=\; u(x')_{2}\,u(x)_{1}\;\widetilde{D}'^{\pm}\,, \qquad\quad \widetilde{u}(x)_{1}\,\widetilde{u}(x')_{2} \;=\; D'^{\pm}\;\widetilde{u}(x')_{2}\,\widetilde{u}(x)_{1}\,, \tag{96}$$

for $\substack{x>x'\\ x<x'}$ and $|x-x'|<2\pi$. They may be established with the help of (81) and (82). The matrix

$$D^{\pm} \;=\; q^{\mp\frac{1}{2}}\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & q^{\pm1}\frac{[p+1]}{[p]} & q^{\mp(p-1)}\frac{1}{[p]} & 0\\ 0 & -q^{\pm(p+1)}\frac{1}{[p]} & q^{\pm1}\frac{[p-1]}{[p]} & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \tag{97}$$

[8]More exactly, they are operator-valued distributions and map into a completion of $\mathcal{H}_{L}$.

represents a special $6j$-symbol[9] dependent on k and j through $q = \mathrm{e}^{-\frac{\pi i}{k+2}}$ and $p = 2j+1$. The matrix $\widetilde{D}^{\pm}$ differs from $D^{\pm}$ by the interchange of the $12,12$ and $21,21$ elements on the diagonal. The primed matrices $D^{'\pm}(p)$ and $\widetilde{D}^{'\pm}(p)$ are equal to the unprimed ones except for $p = k+1$ where, respectively, the third (second) entry on the diagonal is set to zero and for $p = 1$ where the second (third) entry on the diagonal is put to zero.

Let us explain how the latter modifications arise. The relation

$$\chi_{a1}(x)\,\chi_{b2}(x') \;=\; \chi_{b2}(x')\,\chi_{a1}(x)\,D^{\pm}_{1212} \;+\; \chi_{b1}(x')\,\chi_{a2}\,D^{\pm}_{2112}\,, \tag{98}$$

becomes in the action on $\mathcal{V}_{k,j}$ with $j = k+1$ the identity

$$\chi_{a1}(x)\,\chi_{b2}(x') \;=\; =\; \chi_{b2}(x')\,\chi_{a1}(x)\,\frac{q^{\pm\frac{1}{2}}\,[k+2]}{[k+1]} \;-\; \chi_{b1}(x')\,\chi_{a2}(x)\,\frac{q^{\pm(k+\frac{3}{2})}}{[k+1]} \tag{99}$$

which is consistent with vanishing of the raising components $\chi_{a2}(x)$, $\chi_{b2}(x')$ since $[k+2] = 0$. To obtain the commutation relations for the u's, we have to divide both sides of (98) by $[p+1][p]$. This results in the relation

$$u_{a1}(x)\,u_{b2}(x') \;=\; u_{b2}(x')\,u_{a1}(x)\,\widetilde{D}^{\pm}_{1212} \;+\; u_{b1}(x')\,u_{a2}(x)\,\widetilde{D}^{\pm}_{2112}\,. \tag{100}$$

for $p < k+1$. For $p = k+1$, however, the division by $[k+2] = 0$ is not allowed and we have to replace $\widetilde{D}^{\pm}_{1212}(k+1) = q^{\pm\frac{1}{2}}[2]$ by zero to obtain a true relation. The modifications in the exchange relations of $\widetilde{u}$'s follow similarly as do the ones for $p = 1$. In fact the commutation relations on $\mathcal{V}_{k,j'}$ for $j' = \frac{k}{2} - j$ may be obtained from those on $\mathcal{V}_{k,j}$ by simply interchanging the indices $1 \leftrightarrow 2$.

6.3 Quantization of the boundary degrees of freedom

On the quantum group side, following [15], one may realize the representations $\mathcal{V}_{q,j}$ of $\mathcal{U}_q(su(2))$ as a cohomology of modules carrying a representation of an algebra of deformed creators and annihilators $\Pi^{\pm}$, $\Psi^{\pm}$, B, Γ satisfying the relations:

$$\begin{gathered}\Psi\Pi = q\Pi\Psi\,, \qquad \Pi B = B\Pi\,, \qquad \Pi\Gamma = \Gamma\Pi\,, \\ q\,B\Gamma - q^{-1}\Gamma B = q - q^{-1}\,, \quad \Psi B = qB\Psi\,, \quad \Psi\Gamma = q^{-1}\Gamma\Psi\,.\end{gathered} \tag{101}$$

The latter may be represented in the space

$$\mathcal{V} \;=\; \bigoplus_{p \bmod 2(k+2)} \mathcal{V}_{z\,q^p}\,, \tag{102}$$

for z a complex non-zero number, where $\mathcal{V}_{z\,q^p}$ is spanned by the vectors $|\sigma, p\rangle$ with $\sigma = 0, 1, \ldots, k+1$, by setting

$$\begin{gathered}\Pi\,|\sigma,p\rangle = zq^p|\sigma,p\rangle\,, \quad \Psi\,|\sigma,p\rangle = q^{-\sigma}|\sigma,\, p-1 \bmod 2(k+2)\rangle\,, \\ B\,|\sigma,p\rangle = (1-q^{-2\sigma})\,|\sigma-1,p\rangle\,, \quad \Gamma\,|\sigma,p\rangle = (1-\delta_{\sigma,p-1})\,|\sigma+1,p\rangle\,.\end{gathered} \tag{103}$$

[9]The matrix of D corresponds to the lexicographic order of the basis vectors in the tensor product space.

Recall that $\mathcal{U}_q(su(2))$ is generated by $q^{\pm H}$, E and F with the relations

$$q^H E \,=\, E\,q^{H+2}\,, \quad q^H F \,=\, F\,q^{H-2}\,, \quad [E,\,F] \,=\, \frac{q^H - q^{-H}}{q-q^{-1}}\,, \tag{104}$$

which are a deformation of the standard relations for $su(2)$. The spin $j = 0, 1, \ldots, \frac{k+1}{2}$ irreducible highest weight modules $\mathcal{V}_{q,j}$ of $\mathcal{U}_q(su(2))$ have dimension $2j+1$ and are generated from an eigen-vector of $q^{\pm H}$ with eigenvalue $q^{\pm 2j}$ annihilated by E and by F^{2j+1}, the highest weight vector. Upon introduction of the 2×2 matrices built of the $\mathcal{U}_q(su(2))$ generators

$$\gamma \;\equiv\; \begin{pmatrix} q^H & q(q-q^{-1})\,E \\ (q-q^{-1})\,F\,q^H & q(q-q^{-1})^2 F\,E + q^{-H} \end{pmatrix}, \tag{105}$$

the commutation relation (104) may be rewritten in the matrix form

$$\gamma_1 R^+ \gamma_2 (R^\mp)^{-1} \;=\; R^\pm \gamma_2 (R^-)^{-1} \gamma_1\,, \tag{106}$$

which is more convenient for algebraic manipulations. The 4×4 R-matrices are given by

$$R^+ \;=\; q^{\frac{1}{2}} \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 1 & q^{-1}-q & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}, \qquad R^- \;=\; q^{-\frac{1}{2}} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}. \tag{107}$$

Note a useful property: $(R^\pm)^{-1} = P R^\mp P$, where P exchanges the factors in the tensor product.

One may turn the spaces $\mathcal{V}_{z\,q^p}$ into $\mathcal{U}_q(su(2))$-modules by expressing the matrices γ by the deformed creators and annihilators:

$$\gamma \;=\; q \begin{pmatrix} \Pi(1-B\Gamma) & -(\Pi - \Pi^{-1})B + \Pi B \Gamma B \\ -\Pi\Gamma & \Pi^{-1} + \Pi\Gamma B \end{pmatrix}. \tag{108}$$

The commutation relations (104) follow from (101). For z not an (integer) power of q, the $\mathcal{U}_q(su(2))$-modules $\mathcal{V}_{z\,q^p}$ are irreducible. This is in general not the case for z a power of q. To obtain irreducible modules for $z=1$ (it is clearly enough to consider this case), one defines on $\mathcal{V}$ the screening operator $\mathcal{Q} = \Pi\Gamma\Psi^{-2}$ which maps $\mathcal{V}_{q^p}$ to $\mathcal{V}_{q^{p+2}}$ and is nilpotent: $\mathcal{Q}^{k+2} = 0$. It gives rise [15] to the complexes

$$\begin{aligned} &0 \longrightarrow \mathcal{V}_{q^{-p}} \xrightarrow{\mathcal{Q}^p} \mathcal{V}_{q^p} \xrightarrow{\mathcal{Q}^{k+2-p}} \mathcal{V}_{q^{-p}} \longrightarrow 0\,, \\ &0 \longrightarrow \mathcal{V}_{q^p} \xrightarrow{\mathcal{Q}^{k+2-p}} \mathcal{V}_{q^{-p}} \xrightarrow{\mathcal{Q}^p} \mathcal{V}_{q^p} \longrightarrow 0 \end{aligned} \tag{109}$$

of $\mathcal{U}_q(su(2))$-modules exact in the middle. It follows that the action of $\mathcal{U}_q(su(2))$ descends to the cohomology spaces

$$\widetilde{\mathcal{V}}_{q^p} \;=\; \begin{cases} \mathcal{V}_{q^p} / \mathcal{Q}^p(\mathcal{V}_{q^{-p}}) & \text{for} \;\; p = 0, 1, \ldots, k+2\,, \\ \ker \mathcal{Q}^{-p} \subset \mathcal{V}_{q^p} & \text{for} \;\; p = 0, -1, \ldots, -k-2\,. \end{cases} \tag{110}$$

Note that $\widetilde{\mathcal{V}}_1 = \{0\}$ and that $\mathcal{Q}^{k+2-p}$ induces an isomorphism of $\widetilde{\mathcal{V}}_{q^p}$ and $\widetilde{\mathcal{V}}_{q^{-p}}$. For $j = 0, \frac{1}{2}, \ldots, \frac{k+1}{2}$ and $p = 2j+1$, the $\mathcal{U}_q(su(2))$-modules $\widetilde{\mathcal{V}}_{q^p}$ may be identified with the irreducible highest weight modules $\mathcal{V}_{q,j}$ of $\mathcal{U}_q(su(2))$ of spin j with the highest weight vector corresponding to $|0,p\rangle \in \mathcal{V}_{q^p}$. The matrix γ of generators may be decomposed in the action on $\widetilde{\mathcal{V}}_{q^p}$ into the upper- and lower-triangular parts[10] $\gamma_\pm$ such that $\gamma = \gamma_- \gamma_+^{-1}$ with

$$\gamma_+ \;=\; \begin{pmatrix} q^{-\frac{H}{2}} & -(q-q^{-1})q^{-\frac{H}{2}}E \\ 0 & q^{\frac{H}{2}} \end{pmatrix}, \quad \gamma_- \;=\; \begin{pmatrix} q^{\frac{H}{2}} & 0 \\ (q-q^{-1})\,F\,q^{\frac{H}{2}} & q^{-\frac{H}{2}} \end{pmatrix} \tag{111}$$

In terms of $\gamma_\pm$, the commutation relations (106) become

$$\gamma_{+_1}\gamma_{+_2} R^{\pm} \;=\; R^{\pm}\gamma_{+_2}\gamma_{+_1}\,, \quad \gamma_{-_1}\gamma_{-_2} R^{\pm} \;=\; R^{\pm}\gamma_{-_2}\gamma_{-_1}\,, \tag{112}$$

$$\gamma_{+_1}\gamma_{-_2} R^{+} \;=\; R^{+}\gamma_{-_2}\gamma_{+_1}\,, \quad \gamma_{-_1}\gamma_{+_2} R^{-} \;=\; R^{-}\gamma_{+_2}\gamma_{-_1}\,. \tag{113}$$

In order to quantize the monodromy data U, see (76), we shall need also the "quantum group vertex operators", see [15, 22]. Let us introduce two matrices of operators acting in the space $\mathcal{V}$:

$$a \;=\; \begin{pmatrix} -\Psi & qB\Psi \\ \Psi\mathcal{Q} & q(\Pi-\Pi^{-1})\Psi^{-1} - qB\Psi\mathcal{Q} \end{pmatrix} \frac{1}{q-q^{-1}},$$

$$\widetilde{a} \;=\; \begin{pmatrix} -(\Pi-\Pi^{-1})\Psi^{-1} + B\Psi\mathcal{Q} & B\Psi \\ \Psi\mathcal{Q} & \Psi \end{pmatrix}. \tag{114}$$

They satisfy the relations

$$\widetilde{a}\,a \;=\; \tfrac{\Pi-\Pi^{-1}}{q-q^{-1}}\,, \qquad \gamma_1\, a_2 \;=\; a_2\,(R^-)^{-1}\gamma_1\, R^+\,, \qquad \widetilde{a}_1\,\gamma_2 \;=\; R^+\gamma_2\,(R^-)^{-1}\,\widetilde{a}_1\,. \tag{115}$$

The components a_{1a} and $\widetilde{a}_{a2}$ map $\mathcal{V}_{z\,q^p}$ into $\mathcal{V}_{z\,q^{p-1}}$ and, for $z=1$, pass to the quotient spaces (110):

$$a_{1a}:\ \widetilde{\mathcal{V}}_{q^p} \longrightarrow \widetilde{\mathcal{V}}_{q^{p-1}}\,, \qquad \widetilde{a}_{a2}:\ \widetilde{\mathcal{V}}_{q^p} \longrightarrow \widetilde{\mathcal{V}}_{q^{p-1}}\,. \tag{116}$$

Similarly, the components a_{2a} and $\widetilde{a}_{a1}$ map $\mathcal{V}_{z\,q^p}$ into $\mathcal{V}_{z\,q^{p+1}}$ and pass to the quotients:

$$a_{2a}:\ \widetilde{\mathcal{V}}_{q^p} \longrightarrow \widetilde{\mathcal{V}}_{q^{p+1}}\,, \qquad \widetilde{a}_{a1}:\ \widetilde{\mathcal{V}}_{q^p} \longrightarrow \widetilde{\mathcal{V}}_{q^{p+1}}\,. \tag{117}$$

We shall also need modified version of the vertex operators defined by

$$b \;=\; a\,\tfrac{q-q^{-1}}{\Pi-\Pi^{-1}} \tag{118}$$

[10]This requires a choice of the square roots $q^{\pm\frac{H}{2}}$ of $q^{\pm H}$ which may be fixed by demanding that $q^{\pm\frac{H}{2}}|0,p\rangle = q^j|0,2j+1\rangle$ but is irrelevant.

for generic values of z. b is the inverse[11] of the matrix $\widetilde{a}$: $b\,\widetilde{a} = \widetilde{a}\,b = 1$. For $z=1$, b may be still defined on the components $\mathcal{V}_{q^p}$ and $\widetilde{\mathcal{V}}_{q^p}$ such that $[p] \neq 0$. The components of a, $\widetilde{a}$ and b satisfy for generic z the commutation relations which may be summarized as

$$a_1\, a_2 = (\widetilde{D}^{\pm})^{-1}\, a_2\, a_1\, R^{\pm}\,, \quad \widetilde{a}_1\, \widetilde{a}_2 = R^{\pm}\, \widetilde{a}_2\, \widetilde{a}_1\, (D^{\pm})^{-1}\,, \quad b_1\, b_2 = (D^{\pm})^{-1}\, b_2\, b_1\, R^{\pm} \tag{119}$$

with the same p-dependent matrices $D^{\pm}$ and $\widetilde{D}^{\pm}$ as introduced in the previous section, see (97), except for the replacement of $q^{\pm p}$ by the eigenvalues of $\Pi^{\pm}$ equal to $z^{\pm}q^{\pm p}$. For $z=1$ the first two relations of (119) still hold whenever they do not involve division by a vanishing $[p]$, i.e. away from the boundary values $p = 0, k+2$, whereas the 3$^{\rm rd}$ relation requires corrections (as the for the the Kac-Moody fields $u, \widetilde{u}$).

Let us fix $z=1$ and the spins j_0 and j_π between 0 and $\frac{k}{2}$ and define the matrix of monodromy operators

$$\widetilde{U} \;=\; a\,\gamma_{\pi-}^{-1}\gamma_{\pi+}\,\widetilde{a}\,. \tag{120}$$

acting on the space

$$\widetilde{\mathcal{V}} \;=\; \mathop{\oplus}_{p \bmod 2(k+2)} \widetilde{\mathcal{V}}_{q^{p_0}} \otimes \widetilde{\mathcal{V}}_{q^{p_\pi}} \otimes \widetilde{\mathcal{V}}_{q^p} \tag{121}$$

where $p_0 = 2j_0+1$ and $p_\pi = 2j_\pi + 1$, $\gamma_{\pi\pm}$ acts on the $\widetilde{\mathcal{V}}_{q^{p_\pi}}$ factor and a and $\widetilde{a}$ on $\widetilde{\mathcal{V}}_{q^p}$. $\widetilde{U}$ quantizes the classical monodromy matrix U, see (76), modulo potentially singular factors involving $(\Pi - \Pi)^{-1}$ that we have redistributed to the Kac-Moody vertex operators. Note that $\widetilde{U}_{11}$ and $\widetilde{U}_{22}$ preserve the value of p labeling the 3$^{\rm rd}$ representation space $\widetilde{\mathcal{V}}_{q^p}$, $\widetilde{U}_{12}$ lowers it by 2 and $\widetilde{U}_{21}$ raises it by 2 (mod $2(k+2)$). Recall that $\widetilde{\mathcal{V}}_{q^p}$ carries the spin j irreducible representation of $\mathcal{U}_q(su(2))$ for $|p| = 2j+1$ so that the direct sum in (121) contains each spin j factor between 0 and $\frac{k}{2}$ twice and spin $\frac{k+1}{2}$ once.

The tensor products $\widetilde{\mathcal{V}}_{q^{p_0}} \otimes \widetilde{\mathcal{V}}_{q^{p_\pi}} \otimes \widetilde{\mathcal{V}}_{q^p}$ carry the co-product action of $\mathcal{U}_q(su(2))$ defined so that the matrix γ of generators of $\mathcal{U}_q(su(2))$, see (104), acts as $\gamma_{0-}\,\gamma_{\pi-}\,\gamma_w\,\gamma_{\pi+}^{-1}\gamma_{0+}^{-1} \equiv \Delta\gamma$, where γ_w represents the $\mathcal{U}_q(su(2))$-action in the 3$^{\rm rd}$ factor $\widetilde{\mathcal{V}}_{q^p}$. An important property of $\widetilde{U}$, proven in Appendix 1, is that it commutes with the co-product action of $\mathcal{U}_q(su(2))$. It follows then that $\widetilde{U}$ preserves the subspace of the invariant tensors of $\widetilde{\mathcal{V}}$,

$$\widetilde{\mathcal{V}}^{inv} \;=\; \mathop{\oplus}_{p \bmod 2(k+2)} (\widetilde{\mathcal{V}}_{q^{p_0}} \otimes \widetilde{\mathcal{V}}_{q^{p_\pi}} \otimes \widetilde{\mathcal{V}}_{q^p})^{inv} \tag{122}$$

i.e. tensors in the kernel of $(\Delta\gamma - 1)$ where 1 denotes the 2×2 unit matrix. For $0 < j_0, j_\pi, j < \frac{k+1}{2}$, the dimensions of the spaces of invariant tensors are as in the undeformed case [22], i.e. they are given by the standard angular momentum rule:

$$(\widetilde{\mathcal{V}}_{q^{p_0}} \otimes \widetilde{\mathcal{V}}_{q^{p_\pi}} \otimes \widetilde{\mathcal{V}}_{q^p})^{inv} \cong \begin{cases} \mathbf{C} \ \text{if} \ |j_0 - j_\pi| \,\leq\, j \,\leq\, j_0 + j_\pi, \ \ j_0 + j_\pi + j = 0 \bmod 1\,, \\ \{0\} \ \text{otherwise.} \end{cases}$$

[11]In [15], $\widetilde{a}$ was denoted by g_0 and b by g_0^{-1}; the lower indices 1 and 2 of a in (117) and (118) correspond to the upper indices 2 and 1 in [22].

The action of $\widetilde{U}$ on the spaces of invariants may be found by an explicit computation. One obtains for positive p and the diagonal matrix elements of $\widetilde{U}$,

$$\widetilde{U}_{11} \;=\; -\,\frac{q^{-1}(q^{p_0}+q^{-p_0})-q^{p}(q^{p_\pi}+q^{-p_\pi})}{q-q^{-1}}\,, \tag{123}$$

$$\widetilde{U}_{22} \;=\; \frac{q^{-1}(q^{p_0}+q^{-p_0})-q^{-p}(q^{p_\pi}+q^{-p_\pi})}{q-q^{-1}}\,. \tag{124}$$

The non-diagonal matrix elements of $\widetilde{U}$ are for a special choice of the basis of invariant tensors given by:

$$\widetilde{U}_{12} \;=\; q^{-1}(q-q^{-1})\,[j_0+j_\pi+j+1]\,[j_\pi+j-j_0]\,[j_0+j-j_\pi]\,, \tag{125}$$

$$\widetilde{U}_{21} \;=\; q^{-1}(q-q^{-1})\,[j_0+j_\pi-j]\,. \tag{126}$$

The value $p=2j+1$ refers to the tensor in the domain and is lowered by 2 by $\widetilde{U}_{12}$ and raised by $\widetilde{U}_{21}$. The diagonal combinations that are basis independent have the form

$$\widetilde{U}_{12}\,\widetilde{U}_{21} \;=\; q^{-2}(q-q^{-1})^2\,[j_0+j_\pi+j+2]\,[j_\pi+j-j_0+1]\,[j_0+j-j_\pi+1]\,[j_0+j_\pi-j]\,,$$

$$\widetilde{U}_{21}\,\widetilde{U}_{12} \;=\; q^{-2}(q-q^{-1})^2\,[j_0+j_\pi-j+1]\,[j_0+j_\pi+j+1]\,[j_\pi+j-j_0]\,[j_0+j-j_\pi]\,.$$

The space $(\widetilde{\mathcal{V}}_{q^{p_0}}\otimes\widetilde{\mathcal{V}}_{q^{p_\pi}}\otimes\widetilde{\mathcal{V}}_{q^{p}})^{inv}$ is composed of null-vectors of the natural hermitian form if and only if p_0, p_π and p violate the fusion rule $j_0+j_\pi+j\leq k$. Note that $\widetilde{U}_{12}\widetilde{U}_{21}$ vanishes when applied to $(\widetilde{\mathcal{V}}_{q^{p_0}}\otimes\widetilde{\mathcal{V}}_{q^{p_\pi}}\otimes\widetilde{\mathcal{V}}_{q^{p}})^{inv}$ with $j_0+j_\pi+j=k$ due to the vanishing of $[j_0+j_\pi+j+2]=[k+2]$ and, similarly, $\widetilde{U}_{21}\widetilde{U}_{12}$ vanishes for $j_0+j_\pi+j=k+1$. In fact, it is the lowering operator $\widetilde{U}_{12}$ that annihilates $(\widetilde{\mathcal{V}}_{q^{p_0}}\otimes\widetilde{\mathcal{V}}_{q^{p_\pi}}\otimes\widetilde{\mathcal{V}}_{q^{p}})^{inv}$ with $j_0+j_\pi+j=k+1$. The relations for the negative p are a mirror image of those for the positive p so that, for negative p, it is the raising operator $\widetilde{U}_{21}$ that vanishes in the action on $(\widetilde{\mathcal{V}}_{q^{p_0}}\otimes\widetilde{\mathcal{V}}_{q^{p_\pi}}\otimes\widetilde{\mathcal{V}}_{q^{p}})^{inv}$ with $j_0+j_\pi+j=k+1$. These are the only matrix elements that could lead out of the subspace of the null invariant tensors. It follows that $\widetilde{U}_{ij}$ preserve this subspace and, hence, descend to the quotient space on which the hermitian form defines a positive scalar product. The quotient space may be identified with

$$\mathop{\oplus'}_{j}\;(\widetilde{\mathcal{V}}_{q^{p_0}}\otimes\widetilde{\mathcal{V}}_{q^{p_\pi}}\otimes\widetilde{\mathcal{V}}_{q^{p}})^{inv} \tag{127}$$

with $\oplus'$ denoting the sum restricted to j such that $|j_0-j_\pi|\leq j\leq\min(j_0+j_\pi,\;k-j_0-j_\pi)$ and $j_0+j_\pi+j=0\bmod 1$. Note that $j=\frac{k+1}{2}$ is necessarily absent in the sum which falls into two isomorphic terms involving, respectively, the positive and the negative p with no non-zero matrix elements of $\widetilde{U}$ between them. We shall identify the term with positive p with the boundary space of states, see the end of Sect. 6.1:

$$\mathcal{H}^{bd}_{j_0 j_\pi} \;=\; \mathop{\oplus''}_{j}\;(\widetilde{\mathcal{V}}_{q^{p_0}}\otimes\widetilde{\mathcal{V}}_{q^{p_\pi}}\otimes\widetilde{\mathcal{V}}_{q^{p}})^{inv}\,, \tag{128}$$

where the sum $\oplus''$ is as $\oplus'$ but with p restricted to the positive values.

7 Local bulk fields in the boundary WZW $SU(2)$ theory

7.1 Quantization of the classical bulk fields

According to the preceding discussion, see (77), the Hilbert space of the complete boundary theory, combining the bulk Kac-Moody degrees of freedom and the boundary quantum group ones is

$$\mathcal{H}_{j_0 j_\pi} \;=\; \mathop{\oplus''}_{j} \; \mathcal{H}_{j_0 j_\pi j} \;\subset\; \mathcal{H}_L \otimes \mathcal{H}^{bd}_{j_0 j_\pi}\,, \tag{129}$$

where

$$\mathcal{H}_{j_0 j_\pi j} \;\equiv\; \mathcal{V}_{k,j} \otimes (\widetilde{\mathcal{V}}_{q^{p_0}} \otimes \widetilde{\mathcal{V}}_{q^{p_\pi}} \otimes \widetilde{\mathcal{V}}_{q^{p}})^{inv} \tag{130}$$

with $p = 2j+1$. $\mathcal{H}_L$ is given by (90). We are now prepared for quantization of the classical bulk fields $g(t,x)$ defined as functions on the phase space $\mathcal{P}_{\mu_0 \mu_\pi}$ by (54). On the quantum level, we shall define

$$g(t,x) \;=\; u(t+x-2\pi)\, \widetilde{U}\; \widetilde{u}(t-x)\,, \tag{131}$$

see (89) and (91). Note that the matrix elements of $g(t,x)$, acting *a priori* in $\mathcal{H}_L \otimes \mathcal{H}^{bd}_{j_0 j_\pi}$, preserve the diagonal subspace $\mathcal{H}_{j_0 j_\pi}$ and we shall consider them as operators on that space, i.e. on the Hilbert space of the boundary theory. Indeed, in components,

$$g_{ab}(t,x) \;=\; u_{ai}(t+x-2\pi)\, \widetilde{U}_{i\ell}\, \widetilde{u}_{\ell b}(t-x)\,. \tag{132}$$

But $u_{a1}\widetilde{u}_{b1}$ and $u_{a1}\widetilde{u}_{b1}$ preserve the spin j of the Kac-Moody representations, $u_{a1}\widetilde{u}_{b2}$ lowers it by 1 whereas $u_{a2}\widetilde{u}_{b1}$ raises it by 1, just like $\widetilde{U}_{i\ell}$ do for the quantum group spin j. Definition (131) is inspired by the classical formula (54). Quantum fields $g(t,x)$ satisfy the commutation relations that follow from (93), (94) and (92) and encode their current and conformal algebra symmetries:

$$\begin{aligned}
[J(t+x)_1\,,\, g(t,x')_2] \;&=\; 2\pi\,\delta(x-x')\, C_{12}\, g(t,x')_2 \;-\; 2\pi\,\delta(x+x')\, g(t,x')_2\, C_{12}\,,\\
[T(t+x)\,,\, g(t,x')] \;&=\; -\tfrac{3\pi i}{2(k+2)}\, (\delta'(x-x') + \delta'(x+x'))\, g(t,x')\\
&\qquad + 2\pi i\, (\delta(x-x')\, \partial_+ - \delta(x+x')\, \partial_-)\, g(t,x')
\end{aligned}$$

for $|x| \le 2\pi$. The most important property of $g(t,x)$ is the locality:

$$[g(t,x)_1\,,\, g(t,x')_2] \;=\; 0 \qquad \text{for} \quad x \neq x'\,, \tag{133}$$

i.e. the commutativity at different spatial points, a basic property of local quantum field theory considered here on the space time with a finite spatial extension.

7.2 Locality of the bulk fields: generic case

We shall show the locality of $g(t,x)$, the main result of this section, in two steps. First we shall demonstrate that (133) holds on the components of the Hilbert space (129) that do not involve extreme values of spins. This will be done by multiple application of the commutation rules between the building blocks of field g. To perform this calculation, it will be convenient to reshuffle again the $\frac{1}{[p]}$ factors and to introduce a modified version of the monodromy matrix defined for generic values of z labeling the representations of the q-deformed creator and annihilator algebra (101). We set

$$U \;=\; b\, \gamma_{\pi-}^{-1}\gamma_{\pi+}\, \widetilde{a}\,, \tag{134}$$

where b is the matrix of the modified quantum group vertex operators given by (118). The components of U act on the space $\mathcal{V}$ of (102) with generic z. Using the same formula for $z = 1$, one may also define the components $U_{i\ell}$ as operators on the subspace of $\widetilde{\mathcal{V}}$ of (121) with $[p + \delta_{1\ell} - \delta_{\ell 2}] \neq 0$ and one has then the relation

$$U_{i\ell} \;=\; \tfrac{1}{[p+\delta_{i1}-\delta_{i2}]}\,\widetilde{U}_{i\ell} \;=\; \widetilde{U}_{i\ell}\,\tfrac{1}{[p+\delta_{\ell 1}-\delta_{\ell 2}]}\,. \tag{135}$$

to the components of the monodromy matrix $\widetilde{U}$ of (120). We show in Appendix 2 the following commutation relation

$$U_{1}\,(D^{+})^{-1}\,U_{2}\,D^{\mp} \;=\; (D^{\pm})^{-1}\,U_{2}\,D^{-}\,U_{1} \tag{136}$$

holding for generic z and for $z = 1$ whenever no division by vanishing $[p]$ is involved. Note a similarity with the relation (106) with matrices $(D^{\pm})^{-1}$ replacing $R^{\pm}$.

The quantum field (131) may be rewritten with the use of the modified monodromy monodromy U as

$$g(t,x) \;=\; \chi(t+x-2\pi)\,U\,\widetilde{u}(t-x)\,, \tag{137}$$

whenever the right hand side is well defined, see (91). The commutation relations (95) and (96) between the vertex operators χ and $\widetilde{u}$ imply also that, away from the subspaces with vanishing $[p]$,

$$\widetilde{u}(x)_{1}\,\chi(x')_{2} \;=\; \chi(x')_{2}\,(D^{\pm})^{-1}\,\widetilde{u}(x)_{1}\,. \tag{138}$$

Employing this rule, we obtain

$$\begin{aligned} g(t,x)_{1}\; g(t,x')_{2} \;&=\; \chi(t+x-2\pi)_{1}\,U_{1}\,\widetilde{u}(t-x)_{1}\,\chi(t+x'-2\pi)_{2}\,U_{2}\,\widetilde{u}(t-x')_{2} \\ &=\; \chi(t+x-2\pi)_{1}\,U_{1}\,\chi(t+x'-2\pi)_{2}\,(D^{+})^{-1}\,\widetilde{u}(t-x)_{1}\,U_{2}\,\widetilde{u}(t-x')_{2} \end{aligned} \tag{139}$$

or, in components,

$$\begin{aligned} g_{ab}(t,x)\; g_{cd}(t,x') \;&=\; \chi_{ai}(t+x-2\pi)\; U_{i\ell}\; \chi_{cn}(t+x'-2\pi) \\ &\qquad\cdot (D^{+}(\widehat{p}))^{-1}_{\ell n,mr}\; \widetilde{u}_{mb}(t-x)\; U_{rs}\; \widetilde{u}_{sd}(t-x')\,. \end{aligned} \tag{140}$$

Above, $\widehat{p} = 2\widehat{j}+1$ with $\widehat{j}$ equal to the value of spin of the Kac-Moody representation and we reserve the notation $p = 2j+1$ for the labels of the quantum group representations.

In general, the latter are equal to the Kac-Moody ones only in the initial and final states. Taking into account the shifts of the spins, we may rewrite the last relation as

$$g_{ab}(t,x)\; g_{cd}(t,x') \;=\; \chi_{ai}(t+x-2\pi)\; \chi_{cn}(t+x'-2\pi)$$
$$\cdot\, U_{i\ell}\; \left(D^{+}(p+\delta_{1m}-\delta_{2m}+\delta_{1r}-\delta_{2r})\right)^{-1}_{\ell n,mr}\; U_{rs}\; \widetilde{u}_{mb}(t-x)\; \widetilde{u}_{sd}(t-x')\,. \qquad (141)$$

But direct inspection shows that $D^{+}(p+\delta_{1m}-\delta_{2m}+\delta_{1r}-\delta_{2r}) = D^{+}(p)$ (the "ice property" of the $6j$ symbols) and we may use the relation (136) in order to obtain

$$g(t,x)_{1}\; g(t,x')_{2} \;=\; \chi(t+x-2\pi)_{1}\, \chi(t+x'-2\pi)_{2}\; \left(D^{\pm}(p)\right)^{-1} U_{2}$$
$$\cdot\, D^{-}(p)\; U_{1}\; \left(D^{\mp}(p)\right)^{-1}\; \widetilde{u}(t-x)_{1}\; \widetilde{u}(t-x')_{2}\,. \qquad (142)$$

Again the extreme $D^{\pm}(p)$ may be replaced by $D^{\pm}(\widehat{p})$ which permits to use the commutation relations (95) and (96) to infer that

$$g(t,x)_{1}\; g(t,x')_{2} \;=\; \chi(t+x'-2\pi)_{2}\, \chi(t+x-2\pi)_{1}\; U_{2}\; D^{-}(p)\; U_{1}\; \widetilde{u}(t-x')_{2}\; \widetilde{u}(t-x)_{1}$$
$$=\; \chi(t+x'-2\pi)_{2}\; U_{2}\; \chi(t+x-2\pi)_{1}\; D^{-}(p)\; \widetilde{u}(t-x')_{2}\; U_{1}\; \widetilde{u}(t-x)_{1}\,. \qquad (143)$$

Changing back $D^{-}(p)$ to $D^{-}(\widehat{p})$ and applying the commutation relation $\chi(x)_{1}\, D^{\pm}\widetilde{u}(x')_{2} = \widetilde{u}(x')_{2}\, \chi(x)_{1}$ equivalent to (138), we obtain in turn

$$g(t,x)_{1}\; g(t,x')_{2} \;=\; \chi(t+x'-2\pi)_{2}\; U_{2}\; \widetilde{u}(t-x')_{2}\; \chi(t+x-2\pi)_{1}\; U_{1}\; \widetilde{u}(t-x)_{1}$$
$$=\; g(t,x')_{2}\; g(t,x)_{1} \qquad (144)$$

which settles the locality issue for the matrix elements of the field $g(t,x)$ which do not involve the extreme values of spin.

7.3 Locality of the bulk fields: completion of the proof

We still have to look more closely at the commutation relations on the components (129) of the physical Hilbert space close to the extreme values. This will involve a somewhat tedious case by case inspection. Let us rewrite $g(t,x)$ explicitly in terms of the components raising by 1, lowering by 1 and preserving the value of spin j:

$$g_{ab}(t,x) \;=\; g^{+}_{ab}(t,x) \;+\; g^{-}_{ab}(t,x) \;+\; g^{0}_{ab}(t,x)\,, \qquad (145)$$

where

$$g^{+}_{ab}(t,x) = u_{a2}(t+x-2\pi)\, \widetilde{U}_{21}\, \widetilde{u}_{1b}(t-x)\,, \quad g^{-}_{ab}(t,x) = u_{a1}(t+x-2\pi)\, \widetilde{U}_{12}\, \widetilde{u}_{2b}(t-x)\,,$$
$$g^{0}_{ab}(t,x) = u_{a1}(t+x-2\pi)\, \widetilde{U}_{11}\, \widetilde{u}_{1b}(t-x) \;+\; u_{a2}(t+x-2\pi)\, \widetilde{U}_{22}\, \widetilde{u}_{2b}(t-x)\,.$$

The locality statement (133) is equivalent to the equations

$$[g^{+}_{ab}(t,x),\, g^{+}_{cd}(t,x')] \;=\; 0 \;=\; [g^{-}_{ab}(t,x),\, g^{-}_{cd}(t,x')]\,, \qquad (146)$$
$$[g^{+}_{ab}(t,x),\, g^{0}_{cd}(t,x')] \;+\; [g^{0}_{ab}(t,x),\, g^{+}_{cd}(t,x')] \;=\; 0\,, \qquad (147)$$
$$[g^{-}_{ab}(t,x),\, g^{0}_{cd}(t,x')] \;+\; [g^{0}_{ab}(t,x),\, g^{-}_{cd}(t,x')] \;=\; 0\,, \qquad (148)$$
$$[g^{+}_{ab}(t,x),\, g^{-}_{cd}(t,x')] \;+\; [g^{-}_{ab}(t,x),\, g^{+}_{cd}(t,x')] \;+\; [g^{0}_{ab}(t,x),\, g^{0}_{cd}(t,x')] \;=\; 0\,. \qquad (149)$$

We still have to verify the above relations in the action on the spin $j = \frac{k-1}{2}, \frac{k}{2}$ and spin $j = 0, \frac{1}{2}$ components $\mathcal{H}_{j_0 j_\pi j}$ of the Hilbert space (130).

Consider first the case with $j = \frac{k-1}{2}$. In this instant, only the $[g^+, g^+]$ commutation relation of (146) does not follow from the previous considerations but is assured anyway since the combination $g^+ g^+$ vanishes on $\mathcal{H}_{j_0 j_\pi \frac{k-1}{2}}$. The next case $j = \frac{k}{2}$ will require, however, some work. Now the $[g^-, g^-] = 0$ relation of (146) follows from the previous considerations and the $[g^+, g^+] = 0$ one and (147) result from the vanishing on $\mathcal{H}_{j_0 j_\pi \frac{k}{2}}$ of the raising components $\widetilde{u}_{1a}$ of the Kac-Moody vertex operator implying that $g^+ = 0$ on that space. As for the other commutators,

$$
\begin{aligned}
&[g^+_{ab}(t,x),\, g^-_{cd}(t,x')] \\
=\;& \Big(u_{a2}(t+x-2\pi)\,\widetilde{u}_{1b}(t-x)\,u_{c1}(t+x'-2\pi)\,\widetilde{u}_{2d}(t-x') \\
&- u_{c2}(t+x'-2\pi)\,\widetilde{u}_{1d}(t-x')\,u_{a1}(t+x-2\pi)\,\widetilde{u}_{2b}(t-x)\Big)\,\widetilde{U}_{21}\,\widetilde{U}_{12} \\
=\;& -\epsilon_{be}\epsilon_{df}\Big(u_{a2}(t+x-2\pi)\,u_{e2}(t-x)\,u_{c1}(t+x'-2\pi)\,u_{f1}(t-x') \\
&\qquad - u_{c2}(t+x'-2\pi)\,u_{f2}(t-x')\,u_{a1}(t+x-2\pi)\,u_{e1}(t-x)\Big)\,\widetilde{U}_{21}\,\widetilde{U}_{12}\,, \\
&[g^0_{ab}(t,x),\, g^0_{cd}(t,x')] \\
=\;& \epsilon_{be}\epsilon_{df}\Big(u_{a2}(t+x-2\pi)\,u_{e1}(t-x),\, u_{c2}(t+x'-2\pi)\,u_{f1}(t-x') \\
&\qquad - u_{c2}(t+x'-2\pi)\,u_{f1}(t-x'),\, u_{a2}(t+x-2\pi)\,u_{e1}(t-x)\Big)\,\widetilde{U}_{22}^{\,2}
\end{aligned}
\tag{150}
$$

in the action on $\mathcal{H}_{j_0 j_\pi \frac{k}{2}}$. Using the commutation relations (96), it is easy to show that the expressions in the parenthesis vanish implying (149).

As for the commutators in (148), they be may expanded in the same way. It will be instructive to study them for a general values of j to see what changes for the case $j = \frac{k}{2}$. In the action on a general component $\mathcal{H}_{j_0 j_\pi j}$,

$$
\begin{aligned}
&[g^-_{ab}(t,x),\, g^0_{cd}(t,x')] \;+\; [g^0_{ab}(t,x),\, g^-_{cd}(t,x')] \\
=\;& \epsilon_{be}\,\epsilon_{df}\Big[\Big(u_{a1}(t+x-2\pi)\;u_{e1}(t-x)\;u_{c1}(t+x'-2\pi)\;u_{f1}(t-x') \\
&\qquad -u_{c1}(t+x'-2\pi)\;u_{f1}(t-x')\;u_{a2}(t+x-2\pi)\;u_{e1}(t-x)\Big)\,\widetilde{U}_{12}\,\widetilde{U}_{22} \\
&\quad + \Big(-\,u_{a1}(t+x-2\pi)\;u_{e2}(t-x)\;u_{c1}(t+x'-2\pi)\;u_{f1}(t-x') \\
&\qquad +u_{c1}(t+x'-2\pi)\;u_{f2}(t-x')\;u_{a1}(t+x-2\pi)\;u_{e1}(t-x)\Big)\,\widetilde{U}_{11}\,\widetilde{U}_{12} \\
&\quad + \Big(u_{a2}(t+x-2\pi)\;u_{e1}(t-x)\;u_{c1}(t+x'-2\pi)\;u_{f1}(t-x') \\
&\qquad -u_{c2}(t+x'-2\pi)\;u_{e1}(t-x')\;u_{a1}(t+x-2\pi)\;u_{e1}(t-x)\Big)\,\widetilde{U}_{22}\,\widetilde{U}_{12} \\
&\quad + \Big(-\,u_{a1}(t+x-2\pi)\;u_{e1}(t-x)\;u_{c1}(t+x'-2\pi)\;u_{f2}(t-x') \\
&\qquad +u_{c1}(t+x'-2\pi)\;u_{f1}(t-x')\;u_{a1}(t+x-2\pi)\;u_{e2}(t-x)\Big)\,\widetilde{U}_{12}\,\widetilde{U}_{11}\Big].
\end{aligned}
\tag{151}
$$

After reordering with the use of (96) one obtains the expression

$$\epsilon_{be}\,\epsilon_{df}\,\sum_{i=1}^{4} X^{i}_{cfae}\,\Big(T(\widehat{p})_{i1}\,\widetilde{U}_{12}\widetilde{U}_{22} + T(\widehat{p})_{i2}\,\widetilde{U}_{11}\widetilde{U}_{12} + T(\widehat{p})_{i3}\,\widetilde{U}_{22}\widetilde{U}_{12} + T(\widehat{p})_{i4}\,\widetilde{U}_{12}\widetilde{U}_{11}\Big)\,, \quad (152)$$

where

$$\begin{aligned}
X^{1}_{cfae} &= u_{c2}(t+x'-2\pi)\; u_{f1}(t-x')\; u_{a1}(t+x-2\pi)\; u_{e1}(t-x)\,,\\
X^{2}_{cfae} &= u_{c1}(t+x'-2\pi)\; u_{f2}(t-x')\; u_{a1}(t+x-2\pi)\; u_{e1}(t-x)\,,\\
X^{3}_{cfae} &= u_{c1}(t+x'-2\pi)\; u_{f1}(t-x')\; u_{a2}(t+x-2\pi)\; u_{e1}(t-x)\,,\\
X^{4}_{cfae} &= u_{c1}(t+x'-2\pi)\; u_{f1}(t-x')\; u_{a1}(t+x-2\pi)\; u_{e2}(t-x)\,. \qquad (153)
\end{aligned}$$

For $p<k+1$, the matrix $T(p)$ is

$$\begin{pmatrix}
\frac{q^{1\pm1}[p-3]}{[p-1]} & -\frac{q^{-p+2\pm1}[p-3]}{[p-1][p-2]} & \frac{q^{\mp(p-3)}}{[p-2]}-1 & 0\\
-\frac{q^{p-1\pm p\mp1}([p]+[p-2])}{[p-1]^2[p]} & \frac{q^{\pm(p-1)}(1-[p-2]^2)}{[p-1]^2[p-2]}+1 & \frac{q^{p-2\pm1}}{[p-2]} & -\frac{q^{-1\mp1}[p-2]}{[p]}\\
\frac{q^{\pm p\mp1}(1-[p]^2)}{[p-1]^2[p]}-1 & \frac{q^{-p+1\pm p\mp1}([p-2]+[p])}{[p-1]^2[p-2]} & \frac{q^{-1\pm1}[p]}{[p-2]} & \frac{q^{-p\mp1}}{[p]}\\
-\frac{q^{p\mp1}[p+1]}{[p-1][p]} & -\frac{q^{1\mp1}[p+1]}{[p-1]} & 0 & \frac{q^{\mp p\mp1}}{[p]}+1
\end{pmatrix}. \qquad (154)$$

The general commutation relations between $\widetilde{U}_{11}$, $\widetilde{U}_{22}$ and $\widetilde{U}_{12}$ on the space $\mathcal{V}$ of (102) with generic values of z may be read from (136). By continuity, they still hold for $z=1$ (in $\widetilde{U}_{i\ell}$, unlike in $U_{i\ell}$, there no divisions by factors that become singular at $z=1$). They take on $V_{q^{p_0}}\otimes V_{q^{p_\pi}}\otimes V_{q^p}$ the form

$$\widetilde{U}_{12}\widetilde{U}_{22}\,q^{p-1} + \widetilde{U}_{11}\widetilde{U}_{12}\,[p] - \widetilde{U}_{12}\widetilde{U}_{11}\,q^{-1}[p-1] \;=\; 0\,, \qquad (155)$$

$$\widetilde{U}_{12}\widetilde{U}_{22}\,[p-2] - \widetilde{U}_{11}\widetilde{U}_{12}\,q^{-p+1} - \widetilde{U}_{22}\widetilde{U}_{12}\,q^{-1}[p-1] \;=\; 0\,.$$

The vectors $(\widetilde{U}_{12}\widetilde{U}_{22},\,\widetilde{U}_{11}\widetilde{U}_{12},\,\widetilde{U}_{22}\widetilde{U}_{12},\,\widetilde{U}_{12}\widetilde{U}_{11})$ solving (155) are null-vectors of the matrix $T(p)$ assuring the vanishing of the commutator (151).

When $p=k+1$, due to the modification of the commutation relations (96) of the u's (the replacement of the matrix $\widetilde{D}^{\pm}$ by $\widetilde{D}'^{\pm}$), the last column of the matrix $T(p)$ should be replaced by zero. Note that the last line of $T(p)$, unlike the last column, becomes automatically zero for $p=k+1$. It is easy to find then the null-vectors of the resulting 3×3 matrix

$$\begin{pmatrix}
\frac{q^{1\pm1}[4]}{[2]} & \frac{q^{3\pm1}[4]}{[2][3]} & -\frac{q^{\pm1}[4]}{[3]}\\
-q^{-2\mp2} & \frac{q^{\mp1}[4]+1}{[3]} & -\frac{q^{-3\pm1}}{[3]}\\
-1 & \frac{q^{2\mp2}}{[3]} & \frac{q^{-1\pm1}}{[3]}
\end{pmatrix}. \qquad (156)$$

They are proportional to $(1,\, q^{-2},\, q^2{+}1)$ and, if the locality is to hold, so must be the vector $(\widetilde U_{12}\widetilde U_{22},\, \widetilde U_{11}\widetilde U_{12},\, \widetilde U_{22}\widetilde U_{12})$. For $p=k+1$, the relations (155) reduce to the equations

$$-\widetilde U_{12}\widetilde U_{22}\, q^{-2} \,+\, \widetilde U_{11}\widetilde U_{12} \,-\, \widetilde U_{12}\widetilde U_{11}\, q^{-1}[2] \;=\; 0\,,$$

$$\widetilde U_{12}\widetilde U_{22}\,[3] \,+\, \widetilde U_{11}\widetilde U_{12}\, q^{2} \,-\, \widetilde U_{22}\widetilde U_{12}\, q^{-1}[2] \;=\; 0 \tag{157}$$

and imply the desired relation if $\widetilde U_{12}\widetilde U_{11} = 0$. The latter equality fails on the space $V_{q^{p_0}}\otimes V_{q^{p_\pi}}\otimes V_{q^{k+1}}$ but it holds on the quotient space of $(V_{q^{p_0}}\otimes V_{q^{p_\pi}}\otimes V_{q^{k+1}})^{inv}$ by the null invariant tensors. Indeed, the quotient space is non-zero only if the fusion rule $j=\frac{k}{2}\leq \min(j_0+j_\pi,\, k-j_0+j_\pi)$, is satisfied, i.e. if $j_0+j_\pi=\frac{k}{2}$ or $p_0+p_\pi=k+2$. But then $\widetilde U_{11}$ vanishes on $(V_{q^{p_0}}\otimes V_{q^{p_\pi}}\otimes V_{q^{k+1}})^{inv}$ as follows from the explicit formula (123) for its eigenvalue.

The other extreme cases $j=0,\frac{1}{2}$ are similar (in fact, the commutation relations are symmetric under the simultaneous change of spins $j\mapsto\frac{k}{2}-j$ and the interchange of the raising and lowering components of the vertex operators). This ends the proof of locality of the bulk fields $g(t,x)$ in the action on the boundary Hilbert space.

7.4 Boundary 1-point functions of the bulk fields

Given the explicit expressions for the local fields $g(t,x)$ it is not difficult to calculate low-order correlation functions involving these fields, in particular, their 1-point functions given by the matrix elements between the states

$$|j,m\rangle\otimes|p_0,p_\pi,p\rangle \quad\in\quad \mathcal{H}_{j_0 j_\pi}\,, \tag{158}$$

see (129,130), where $|j,m\rangle$ is the state in $\mathcal{V}_{k,j}$ annihilated by the positive current modes J_n and such that $(J^3_0-m)\,|j,m\rangle=0$ and $|p_0,p_\pi,p\rangle$ is in the (one-dimensional space) $(\widetilde{\mathcal V}_{q^{p_0}}\otimes\widetilde{\mathcal V}_{q^{p_\pi}}\otimes\widetilde{\mathcal V}_{q^{p}})^{inv}$ with $p=2j+1$. The structure (132) of fields g reduces this calculation to that of the matrix elements of $\widetilde U$ given above by Eqs. (123,124,125,126). It remains then to calculate the matrix elements

$$\langle j,m'\mid u_{ai}(t+x-2\pi)\,\widetilde u_{\ell b}(t-x)\mid j,m\rangle\,, \tag{159}$$

where $j'=j$ for $i=\ell=1$ or $i=\ell=2$, $j'=j+1$ for $i=2$, $\ell=1$ and $j'=j-1$ for $i=1$, $\ell=2$. This is, in fact, a special conformal 4-point block whose computation may be found in [25]. Clearly, the expression (159) is time independent so that we may set $t=0$. It is more convenient to use the exponential coordinate. Set $\ln z=i(x-2\pi)$ and $\ln\widetilde z=-ix$. The conformal block corresponds to insertion of the primary field labeled of spin j at zero, spin j' at infinity and spin $\frac{1}{2}$ at z and $\widetilde z$, see Fig. 5.

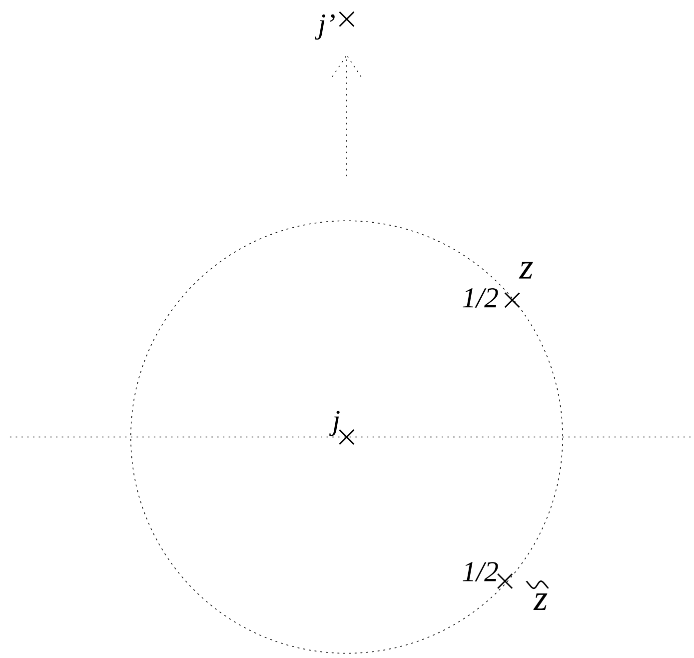

Fig. 5

The conformal invariance imposes the form

$$(iz)^{\Delta_{1/2}}\,(i\widetilde{z})^{\Delta_{1/2}}\,z^{\Delta_{j'}-2\Delta_{1/2}-\Delta_j}\;F_{i\ell}^{m,a,b,m'}(\eta) \tag{160}$$

where $\Delta_j = \frac{j(j+1)}{k+2}$ is the conformal weight of the spin j primary field and $\eta = \widetilde{z}/z$. In its dependence on (m', a, b, m), $F_{i\ell}$ is an invariant tensor in the tensor product $\mathcal{V}_{j'} \otimes \mathcal{V}_{1/2} \otimes \mathcal{V}_{1/2} \otimes \mathcal{V}_j$ of four representations of the $SU(2)$ group and it satisfies the Knizhnik-Zamolodchikov equation

$$\left[(k+2)\frac{d}{d\eta} + \frac{C_{23}}{1-\eta} - \frac{C_{34}}{\eta}\right] F_{i\ell}(\eta) \;=\; 0\,, \tag{161}$$

where C_{23} and C_{34} are the Casimir operator of $su(2)$ acting in the product of the two middle (spin 1/2) representations and of the last two ones, respectively.

When $j' = j$, there are two linearly independent $SU(2)$ invariants I_0 and I_1 and they may be chosen so that

$$\begin{aligned} C_{23}\,I_0 \;&=\; \tfrac{1}{2}\,I_0 + I_1\,, \qquad & C_{23}\,I_1 \;&=\; -\tfrac{3}{2}\,I_1\,, \\ C_{34}\,I_0 \;&=\; -(j+1)\,I_0\,, \qquad & C_{34}\,I_1 &= I_0 + j\,I_1\,. \end{aligned} \tag{162}$$

Eq. (161) reduces to the hypergeometric one and two solutions corresponding to F_{11} and F_{22} are [25]

$$F_{11}(\eta) \;=\; \frac{n_{11}(j)\;\eta^{\frac{j}{k+2}}}{(1-\eta)^{\frac{3}{2(k+2)}}}\Big[\,2j\,(1-\eta)\;F(1-\tfrac{1}{k+2}, 1+\tfrac{2j}{k+2}, 1+\tfrac{2j+1}{k+2};\,\eta)\;I_0$$

$$+\,(2j+1)\ F(-\tfrac{1}{k+2},\tfrac{2j}{k+2},\tfrac{2j+1}{k+2};\,\eta)\ I_1\Big]\,,$$

$$F_{22}(\eta)\ =\ \frac{n_{22}(j)\ \eta^{-\frac{j+1}{k+2}}}{(1-\eta)^{\frac{3}{2(k+2)}}}\Big[(k+1-2j)\,(1-\eta)\ F(1-\tfrac{1}{k+2},1-\tfrac{2j+2}{k+2},1-\tfrac{2j+1}{k+2};\,\eta)\ I_0$$
$$-\,\eta\ F(1-\tfrac{1}{k+2},1-\tfrac{2j+2}{k+2},2-\tfrac{2j+1}{k+2};\,\eta)\ I_1\Big] \qquad (163)$$

behaving when $\eta\to 0$ as $\mathcal{O}(\eta^{\Delta_{j\pm 1/2}-\Delta_{1/2}-\Delta_j})$, respectively. The commutation relations (138) with the D-matrices (97) fix the ratio $\frac{n_{22}(j)}{n_{11}(j)}=(2j+1)\,\frac{\Gamma(\frac{2j+1}{k+2})\,\Gamma(1-\frac{2j}{k+2})}{\Gamma(1-\frac{2j+1}{k+2})\,\Gamma(\frac{2j+2}{k+2})}$.

When $j'=j\pm 1$, up to normalization, there is only one invariant tensor $J_\pm$ in the space $\mathcal{V}_{j'}\otimes\mathcal{V}_{1/2}\otimes\mathcal{V}_{1/2}\otimes\mathcal{V}_j$ and the Kznizhnik Zamolodchikov equation takes the scalar form

$$0\ =\ \left[\frac{d}{d\eta}+\frac{\Delta_1-2\Delta_{1/2}}{1-\eta}-\frac{\Delta_{j\pm1/2}-\Delta_{1/2}-\Delta_j}{\eta}\right]\begin{cases}F_{21}(\eta)\\ F_{12}(\eta)\end{cases} \qquad (164)$$

with the solutions

$$F_{12}=n_{12}(j)\ \eta^{\Delta_{j-1/2}-\Delta_{1/2}-\Delta_j}\,(1-\eta)^{\Delta_1-2\Delta_{1/2}}\,J_-=n_{12}(j)\ \eta^{-\frac{j+1}{k+2}}\,(1-\eta)^{\frac{1}{2(k+2)}}\,J_-\,,$$
$$F_{21}=n_{21}(j)\ \eta^{\Delta_{j+1/2}-\Delta_{1/2}-\Delta_j}\,(1-\eta)^{\Delta_1-2\Delta_{1/2}}\,J_+=n_{21}(j)\ \eta^{\frac{j}{k+2}}\,(1-\eta)^{\frac{1}{2(k+2)}}\,J_+\,. \qquad (165)$$

The normalizations $n_{i\ell}(j)$ could be constraint further using the explicit free field realizations of the operators χ and $\widetilde{u}$.

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Appendix 1

We prove here that the monodromy operator $\widetilde{U}$ commutes with the co-product action of $\mathcal{U}_q(su(2))$ in the space (121). We have to show that

$$\Delta\gamma_{_1}\,\widetilde{U}_{_2}\ =\ \widetilde{U}_{_2}\,\Delta\gamma_{_1}\,. \qquad (1)$$

This is follows with the use of the commutation relations (112) and (115). Indeed,

$$(\gamma_{0-}\,\gamma_{\pi-}\gamma_w\,\gamma_{\pi+}^{-1}\gamma_{0+}^{-1})_1\ (a\,\gamma_{\pi-}^{-1}\gamma_{\pi+}\,\widetilde{a})_2$$

$$= (\gamma_{0-}\gamma_{\pi-})_1 (\gamma_w)_1 \, a_2 (\gamma_{\pi+}^{-1})_1 (\gamma_{\pi-}^{-1})_2 \, (\gamma_{\pi+}\widetilde{a})_2 \, (\gamma_{0+}^{-1})_1$$

$$= (\gamma_{0-}\gamma_{\pi-})_1 \, a_2 (R^-)^{-1} (\gamma_w)_1 \, (\gamma_{\pi-}^{-1})_2 \, (\gamma_{\pi+}^{-1})_1 R^+ (\gamma_{\pi+})_2 \, \widetilde{a}_2 \, (\gamma_{0+}^{-1})_1$$

$$= a_2 \, (\gamma_{0-})_1 \, (\gamma_{\pi-})_1 (R^-)^{-1} \, (\gamma_{\pi-}^{-1})_2 \, (\gamma_w)_1 (\gamma_{\pi+})_2 R^+ \, (\gamma_{\pi+}^{-1})_1 \, \widetilde{a}_2 \, (\gamma_{0+}^{-1})_1$$

$$= a_2 \, (\gamma_{0-})_1 \, (\gamma_{\pi-}^{-1})_2 (R^-)^{-1} \, (\gamma_{\pi-})_1 \, (\gamma_{\pi+})_2 \, (\gamma_w)_1 R^+ \, \widetilde{a}_2 \, (\gamma_{\pi+}^{-1}\gamma_{0+}^{-1})_1$$

$$= (a\,\gamma_{\pi-}^{-1})_2 \, (\gamma_{0-})_1 \, (\gamma_{\pi+})_2 \, (\gamma_{\pi-})_1 \, \widetilde{a}_2 \, (\gamma_w \, \gamma_{\pi+}^{-1}\gamma_{0+}^{-1})_1$$

$$= (a\,\gamma_{\pi-}^{-1}\gamma_{\pi+}\widetilde{a}_2)_2 \; (\gamma_{0-}\,\gamma_{\pi-}\,\gamma_w\,\gamma_{\pi+}^{-1}\gamma_{0+}^{-1})_1 \,. \tag{2}$$

Appendix 2

We prove here the commutation relation (136).

$$U_1 \, (D^+)^{-1} \, U_2 \;=\; (b\,\gamma_{\pi-}^{-1}\,\gamma_{\pi+}\,\widetilde{a})_1 \, (D^+)^{-1} \, (b\,\gamma_{\pi-}^{-1}\,\gamma_{\pi+}\,\widetilde{a})_2$$

$$= (b\,\gamma_{\pi-}^{-1}\,\gamma_{\pi+})_1 \, b_2 \, (R^+)^{-1} \, \widetilde{a}_1 \, (\gamma_{\pi-}^{-1}\,\gamma_{\pi+}\,\widetilde{a})_2$$

$$= b_1 \, b_2 \, (\gamma_{\pi-}^{-1}\,\gamma_{\pi+})_1 \, (R^+)^{-1} \, (\gamma_{\pi-}^{-1}\,\gamma_{\pi+})_2 \, \widetilde{a}_1 \, \widetilde{a}_2$$

$$= (D^\pm)^{-1} \, (b_2 \, b_1 \, R^\pm \, (\gamma_{\pi-}^{-1})_1 \, (\gamma_{\pi-}^{-1})_2 \, (R^+)^{-1} \, (\gamma_{\pi+})_1 \, (\gamma_{\pi+})_2 \, R^\mp \, \widetilde{a}_2 \, \widetilde{a}_1 \, (D^\mp)^{-1}$$

$$= (D^\pm)^{-1} \, b_2 \, b_1 \, (\gamma_{\pi-}^{-1})_2 \, (\gamma_{\pi-}^{-1})_1 \, R^- \, (\gamma_{\pi+})_2 \, (\gamma_{\pi+})_1 \, \widetilde{a}_2 \, \widetilde{a}_1 \, (D^\mp)^{-1}$$

$$= (D^\pm)^{-1} \, (b\,\gamma_{\pi-}^{-1})_2 \, b_1 \, (\gamma_{\pi+})_2 \, R^- \, (\gamma_{\pi-}^{-1})_1 \, \widetilde{a}_2 \, (\gamma_{\pi+}\,\widetilde{a})_1 \, (D^\mp)^{-1}$$

$$= (D^\pm)^{-1} \, (b\,\gamma_{\pi-}^{-1}\gamma_{\pi+})_2 \, b_1 \, R^- \, \widetilde{a}_2 \, (\gamma_{\pi-}^{-1}\,\gamma_{\pi+}\,\widetilde{a})_1 \, (D^\mp)^{-1}$$

$$= (D^\pm)^{-1} \, (b\,\gamma_{\pi-}^{-1}\gamma_{\pi+}\,\widetilde{a})_2 \, D^- \, (b\,\gamma_{\pi-}^{-1}\,\gamma_{\pi+}\,\widetilde{a})_1 \, (D^\mp)^{-1} \;=\; (D^\pm)^{-1} \, U_2 \, D^- \, U_1 \, (D^\mp)^{-1} \,.$$

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